Original Article

The Process of Constructing Speed-time Function from Distance-Time Function $y=2^x$ by Students Using Different Reasoning Methods for Continuous Change

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연속적인 변화에 대한 추론 방식이 다른 학생들의 거리함수 $y=2^x$ 에서 속력함수를 구성하는 과정에 대한 연구

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Abstract

This study investigates the way students perceive and express change in an exponential situation with different reasoning methods for continuous change - for example, students, who use chunky reasoning, construct a new function representing the change of exponential distance-time function. Using case study approach, the study presents the results of teaching experiments conducted on three high school freshmen students. It may be difficult to generalize the results since the sample size was limited; however, the findings have the potential to inform researchers in mathematics education about students' conceptual knowledge of continuous change. The information about the process of constructing the speed-time function from the distance-time function based on the students' understanding of continuous change may also be helpful in interpreting student thinking in differential concept.

Keywords: exponential function, quadratic function, distance-time function, speed-time function, chunky reasoning, smooth reasoning

본 연구는 덩어리 추론을 하는 학생들이 지수함수적 상황의 거리함수에서 어떻게 변화를 나타내는 새로운 함수를 구성하는 지에 대한 사례를 제시한 연구이다. 본 연구는 연속적인 변화에 대한 추론 방식의 차이가 있는 학생들을 대상으로 지수함수적 상황에서 학생들이 연속적인 변화를 어떻게 인식하고 표현하는지에 대하여 주목하고 고등학교 1학년 학생 세 명을 대상으로 20차시 교수실험을 진행한 결과를 담고 있다. 본 연구에서의 결과에 따르면 덩어리 추론을 하는 학생이 지수함수적 상황의 거리함수의 그래프에서 소구간 마다의 평균변화율을 함수값으로 가지는 대수식을 구성한 다음, 소구간의 폭을 줄여감에 따라 소구간에 대응된 평균변화율의 변화를 관찰하여 속력함수의 대수식을 구성하는 장면이 관찰되었다. 제한된 학생들을 대상으로 진행된 교수실험의 결과를 일반화하여 바로 학교수학에 적용하는 것에는 어려움이 있겠지만, 학생들의 연속적인 변화에 대한 이해의 필요성을 드러낼 수 있다는 측면에서 본 연구가 수학교육에 기여하는 바가 있을 것으로 판단된다. 또한 학생들의 연속적인 변화에 대한 이해에 근거하여 거리함수에서 속력함수를 구성하는 과정에 대한 정보는 미분 개념 관련 연구에도 도움이 될 것으로 보인다.

핵심어: 지수함수, 이차함수, 거리함수, 속력함수, 덩어리 추론, 매끄러운 추론

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I. INTRODUCTION

Carlson et al. (2002) stated the psychological activity of recognizing the change and making relational adjustments between two different quantities as covariational reasoning. In other words, covariational reasoning is a mental activity in which one infers the relation between two different quantities when the two change in relation to one another, and it is considered important when understanding the relation between two dynamically changing variables. In this context, how students perceive and describe the changing relation between continuous variations in dynamics can serve as a significant baseline data for studies of covariational reasoning.

To examine students' perception and expression, the study can be divided into two sections: 1) study of degree of change and 2) study of reasoning the change.

Study of degree of change is about distinguishing the magnitude of change in terms of 'quantity of change' and 'intensity of change'. 'Quantity of change' involves perceiving the change solely through the difference in functional values, whereas 'intensity of change' involves perceiving the degree of change by considering the difference in both functional values and domain.

Meanwhile, research done by Castillo-Garsow (2012) can be considered as a study which regards the methods for reasoning the change. Castillo-Garsow (2012) explained the perception for continuous change with smooth reasoning and chunky reasoning, where smooth reasoning is a method which infers the continuous change as whole by changes at instantaneous moments, and chunky reasoning is a method to split the intervals and infer the whole change from the divided chunks. In this sense, smooth reasoning and chunky reasoning can have distinct ways of describing continuous change and their mathematical output can also be different (Kim & Shin, 2016)¹⁾.

Recent studies of teaching experiment with given task handling dynamics (Lee, 2017a; Lee et al., 2016; Lobato et al., 2012; Thompson, 1994) which deals with the question 'how does a student perceive and express change in relation to two continuously changing variables' can also be interpreted in the same way. An object's movement can be expressed with continuous change of time, speed, and distance along with the relations of each variable. It is also notable that the history of the differentiation has close relation to concerns for 'dynamics at an instant' or 'instantaneous change', which is why Gravemeijer & Doorman (1999) saw the modeling for velocity and distance relation for dynamics as a starting point for differential and integral calculus.

There are also studies done in Korea which examines students' perception and expression for continuous change through distance-time function and speed-time function. Study done by Lee (2017b) reveals that for cubic distance-time function and speed-time function, not only did students perceive

1) The study expresses the presentation of mathematical results as 'construct', because the study focuses on the effect of students' reasoning methods on future learning outcomes. Similarly, this study will also express the process of deriving the mathematical result as 'process of construct'.

the idea of average speed as a new quantity, they treated average speed as a changing quantity and constructed a function for average speed and time; they used this function to derive speed-time function from distance-time function. These previously referred studies deal with polynomial functions such as quadratic or cubic function; there are also studies which discuss changes in exponential functions.

Confrey & Smith (1994) compared quadratic function $y=x^2$ and exponential function $y=2^x$ and used the term 'additive reasoning' and 'multiplicative reasoning' to distinguish the apparent change of the two function, and Ellis (2011) further took the research and suggested a 'shift' for multiplicative reasoning in case of exponential function. Both quadratic and exponential functions differ from linear function in that their rate of change changes. Therefore, in order to distinguish quadratic and exponential functions from each other, one needs to perceive the quantity of change as a distinct value and compare the value's change. Research done by Confrey & Smith (1994) and Ellis (2011) have its significance in their approach of comparing quadratic and exponential functions and explaining the change of exponential function by relating it to multiplicative reasoning. Meanwhile Lee, Yang, & Shin (2017) directly showed examples of students' perception and constructing procedure for particular momentary change in exponential function. The research conducts an experiment where students with no learning experience of differentiation are given the distance-time function $y=2^x$ and asked to derive instantaneous speed at $x=0$. But the study shows that students were not able to derive speed-time function for distance-time function $y=2^x$, and the instantaneous speed at $x=0$ was not derived from the viewpoint of change or rate of change; it was derived in somewhat the same way they previously used to construct speed-time function for polynomial distance-time function.

To sum up, the way students perceive continuous change can help research on covariational reasoning which is the reasoning of relation between two variables; perceiving change through notion of intensity of change can contribute to study regarding students' development on the concept of differentiation. Also, studies showed that students were able to use either smooth reasoning or chunky reasoning when they are trying to reason with continuous change; there were also researches examining student's way of understanding apparent changes in exponential function, which is well distinct from apparent change of quadratic function. This study further examines how students perceive and express changes in exponential function by conducting teaching experiments on students who differ in their reasoning of continuous change. Particularly, this is a study proposing a case for 'how do students using chunky reasoning construct a function representing the change of exponential distance-time function'. It would be difficult to apply the results directly to mathematics in school fields, for it is a teaching experiment conducted with a limited number of students, but it is expected for this research to have its significance on revealing the need for studies on students' comprehension of continuous change.

This research discusses only a portion of 20 teaching experiments (16th, 17th, 18th, 19th and 20th) which are related to the subject of the study, especially focusing on 19th and 20th teaching experiment to suggest a viewpoint for the following research problem

· Among the three students with different reasoning methods for continuous change, how does student C, who uses chunky reasoning, constitute the speed-time function from the exponential distance-time function $y=2^x$?

II. LITERATURE REVIEW

1. Continuous change reasoning

There are studies presenting direct or indirect information on how students act to perceive continuous change in covariational relation (Confrey & Smith, 1994; Ellis, 2012; Lee & Shin, 2017; Lobato et al., 2012). These studies show that students substitute gradually increasing discrete values (such as 1, 2, 3, ...) from the domain and form correspondence table from the function values obtained. Table 1 is an example of correspondence table for understanding change in covariational relation for function $y=x^2$

This method agrees with the suggestion Carlson et al. (2002) made that in covariational reasoning, people understand and handle covariational relation in a discrete way.

However, Saldanha & Thompson (1998) pointed out that this method only treats changes of discrete values and not the changes in between; they argued that it is not an appropriate way to perceive change in continuous situations.

In students' reasoning with continuous change, Castillo-Garsow (2012) suggested two separate methods of smooth reasoning and chunky reasoning. For example, when asked to calculate the total sum of license

Table 1 Example of correspondence table to understand changes in covariational relation for function $y=x^2$

x	1	2	3	...
y	1	4	9	...

Table 2 Graph results for smooth reasoning and chunky reasoning on task of calculating total sum of TV licence fee

fee for 10 months when the license fee of TV is 2000 won every first day of the month, the students displayed dissimilar results. The responses were 1) perceiving the time change as discrete variation and adding 2000 won as every month changes, 2) still perceiving time discretely but expressing monthly elapsed time in x -coordinate and expressing the accumulated license fee as y to plot the points and connect them to form a continuous line ascending to the right side 3) drawing the graph in a discontinuous stair figure. To draw the results, it is as it's shown on *Table 2*

First and second group shows that when the students perceived the continuous change of time they only considered the changes at the ends of each interval. More importantly, as they were comprehending the change as a whole, they did not consider the time between '1st and 2nd month' or '2nd and 3rd month' which correspond to sections between the points. In contrast, the 3rd group with a staircase graph can be treated as having considered the changes inside the interval. Castillo-Garsow (2012) categorized the first and second response as chunky reasoning and third response as smooth reasoning in regard for students' perception and comprehension of the change.

Meanwhile Lee & Shin (2017) made an observation that when students were given a graph displaying function with continuous change, the way chunky reasoning students explain the graph's change are rather similar to that of smooth reasoning students, but chunky reasoning students have more diverse ways of explanation by the way they split the interval into smaller sections. In an experiment, both students using chunky reasoning smooth reasoning were given distance-time function $y = \sqrt{x}$ and asked to interpret the change by using the graph. All students expressed that "object's speed following the distance-time function $y = \sqrt{x}$ gradually slows down" when describing the change. But when it came to explaining their interpretation through graphs the students differed; following are the expressions three students used on their explanation:

S1: The lines connecting the origin and the graph are decreasing in slope.

S2: The lines connecting each end of the graph on small intervals are decreasing in slope

S3: Tangential lines on every point are decreasing in slope

Figure 1 is rearranged model of what the three students demonstrated while explaining their method.

Figure 1 Reference graphs of each student explaining the change of distance-time function $y = \sqrt{x}$

According to Castillo-Garsow (2012)'s method, S1 and S2 infers continuous change through dividing intervals into small sections and therefore can be treated as students using chunky reasoning. S3, on the other hand, perceives change on every instant when inferring continuous change, which can well be treated as a student using smooth reasoning. Moreover, chunky reasoning students S1 and S2 differed in ways of splitting when explaining continuous change through graph; the difference shown can be interpreted from Lobato et al. (2012)'s study confronting students' perception for change in cumulative way and sectional way.

Through what has been presented so far, by ways of perceiving, students' inference regarding continuous change can be divided into chunky reasoning and smooth reasoning (Castillo-Garsow, 2012), and by methods of splitting intervals, chunky reasoning can be further divided into cumulative division and sectional division (Lee & Shin, 2017; Lobato et al., 2012).

2. A case on the composition of the speed-time function in the distance-time function

Research on the relation between time, speed, and distance in dynamics were important subjects in conceptual development of function and limit. Amongst other relations representing dynamics, time-wise relation to other variables was especially noted. Often mentioned examples are distance-time function and speed-time function; perception of 'degree of change' for both functions and perception of 'relation to one another' provide crucial significance in developmental study of differential and integral calculus.

Some researchers in Korea have conducted research on the expression of students facing tasks dealing with relations between time, speed, and distance in dynamics (Lee, Moon, & Shin, 2015; Lee et al., 2016; Lee, 2017a; Lee, 2017b; Lee, Yang, & Shin, 2017; Lee & Shin, 2017). These studies point to the fact that research on reasoning about continuous change are related to perception of instantaneous change, providing information about how students with different methods of reasoning for continuous change perceive instantaneous change in distance-time function and speed-time function.

Lee et al. (2015) describes the changes in students' expression of the change in the graph of function $y=x^2$. At the beginning, students constructed a correspondence table, drawing the points on the coordinate plane using the results, and smoothly connecting the points to construct graph of function $y=x^2$. In this process, the students used the expression "the value of y increases faster as the value x increases" to the change of graph of function $y=x^2$. At this point, the researcher asked how the graph of the function $y=2^x$ changes; the students again used the expression "the value of y increases faster as the value x increases" at first, but they naturally proceeded to distinguish the difference between the change of previously given function $y=x^2$.

The students were then given the task of determining the magnitude of change on intervals $[1,2]$ and $[2,2.5]$ of the function $y=x^2$ and were asked to tell which one is greater. Some students responded that the change on interval $[1,2]$ was greater because $f(2) - f(1) > f(2.5) - f(2)$, and some students responded

that the degree of change in the interval $[2, 2.5]$ was greater because $\frac{f(2.5) - f(2)}{2.5 - 2} > \frac{f(2) - f(1)}{2 - 1}$. However, through communication, the three students agreed that it was more appropriate to explain the change by $\frac{f(2.5) - f(2)}{2.5 - 2} > \frac{f(2) - f(1)}{2 - 1}$, and ended up explaining that one can extend the method to express the change of the function $y=x^2$ on an interval by the slope of the line segment connecting the two endpoints of the given interval. Also, by using this method, students divided the domain of the function $y=x^2$ into intervals of '0 to 1', '1 to 2', '2 to 3', ..., and calculated the slope of the line segment of each corresponding intervals; subsequently, the presented a staircase graph by using the relation between the intervals and their representative change. Finally, the students repeated the process of observing the interval by cutting it in half. Through the process of comparing the obtained staircase graphs, the students found that the staircase graph representing the change of the function $y=x^2$ becomes a straight line when the width of the interval is reduced. This can be seen as a result corresponding to the derivative of the function $y=x^2$ (Figure 2).

Following the study by Lee et al. (2015), the researcher conducted 20 sessions of teaching experiment on three 10th grade high school students who had no experience of learning differentiation. The three students participating in the study all had different reasoning for continuous change according to the criteria of Castillo-Garsow (2012) and Lee & Shin (2017). S1 is a student of chunky reasoning who observes change in a cumulative way, S2 is a student using chunky reasoning who observes the change on a segmental basis, and S3 is a smooth reasoning student who expresses the change as the slope of the tangent at the point. Lee et al. (2016) showed students' perception of the relations between time, speed, and distance based on their teaching experiment data. In perception of the relation between time, speed, and distance, the students first saw the relation in a way that, if two values are given, one can get the other remaining value. One being able to obtain the other value when the two values are given.

Lee (2017a), on the other hand, contains the contents of ongoing teaching experiments following the study of Lee et al. (2016). The research describes how students who initially constructed distance-time function as $y=x^2$ from the speed-time function $y=x$ change to constructing $y=\frac{x^2}{2}$ for distance-time function

Figure 2 Configuration steps of the staircase graph representing the change of quadratic function

after the change of expression for speed (change from treating it as substitutional value into having expressional distinction of ‘average of varying speed’ and ‘speed at an instant’). In particular, Lee (2017a) describes the process of constructing the speed-time function $y=x^2$ from the distance-time function $y=\frac{x^3}{3}$ based on the knowledge that the area below the graph of speed-time function corresponds to the total distance traveled by an object, which was previously known to the students: in doing so, the students again changed the expression of “average of changing speeds” to “average speed”. In addition, students who used different reasoning for continuous change each showed a difference in the expression of average speed, and they perceived the average speed as a changing quantity and constructed a relation of ‘average speed function’. What should be noted here is that the task to construct the average of varying speeds is a complex task which deals with continuous variables along with discrete concept of average; divergence in reasoning methods for continuous change also led to difference in mathematical results. When confronted with the task to defining an average of continuously varying speed, students using chunky reasoning figured that the average should be acquired through discrete data; they divided the section into smaller segments to obtain the average speed in the small sections, and again calculated the average of the obtained values. On the other hand, although the concept of average meant adding up discrete data and dividing them, the student with smooth reasoning pointed out the fact it is an average of continuously changing speed; the student perceived the average speed as the height of a rectangle which has the same area as the lower part of the speed-time function, where the width of the rectangle equals to the time interval (Figure 3).

In addition, difference between chunky reasoning and smooth reasoning students was observed when

Figure 3 Illustration of student using smooth reasoning to imply meaning of average speed between 0 to 4 sec-
ond when the object’s speed-time function is $y=2^x$

constructing average speed of distance-time function $y = \frac{x^3}{3}$ from time a to k and then making adjustment of $a \rightarrow k$. When given distance-time function was $y=f(x)$, both students constructed the average speed in the interval $a \rightarrow k$ as $\frac{f(k)-f(a)}{k-a}$. However, in the later process, the student using smooth reasoning reduced $(k-a)$ in the denominator and substituted $a=k$ to obtain the result. For example, as for the distance-time function $y = \frac{x^3}{3}$, the student first expressed the average speed in the interval $[a,k]$ as $\frac{\frac{k^3}{3} - \frac{a^3}{3}}{k-a}$ and then reduced the factor $(k-a)$ in the denominator to construct $\frac{k^2 + ak + a^2}{3}$, and substituted $a=k$ to obtain k^2 . The student also gave explanation to the result of average speed turning into instantaneous speed, which is as follows: "Since $\frac{f(k)-f(a)}{k-a}$ means the average speed from time a to k , $\frac{f(a)-f(a)}{a-a}$ should mean the average speed from time a to a , which consequently infers the speed at the moment a ". The student used this result of instantaneous speed being k^2 at time k to construct the speed-time function of $y=x^2$, based on his own understanding that a function which has the instantaneous speed as the function value is a speed-time function.

3. Changes in exponential situation

After Confrey & Smith (1994) studied the multiplicative rate of change in the exponential function, there were discussions made either directly or indirectly by studies such as Thompson (2008) and Ellis (2011) about the multiplicative rate of change. The beginning of these studies is the part Confrey & Smith (1994) explained the change in the quadratic function as an additive rate of change and the change in the exponential function as a multiplicative rate of change when comparing changes in quadratic and exponential functions. The notable point here is that in order to see the change in the exponential situation, the change was compared with that of the quadratic function situation. The study of Lee, Moon & Shin (2015) also involves a scene that looks at changes in exponential situations based on changes in quadratic situations, which can also be understood in the same context. It is meaningful in that it allows you to think about the method of introducing the derivative of the exponential function as the concept of the instantaneous rate of change using the limit of the average rate of change.

Until now, the concept of rate of change in learning of exponential function was taught by deriving the concept of instantaneous rate of change from the limit of average rate of change in conjunction with calculus learning, and then extending to defining the derivative having the instantaneous rate of change as the function value. If the average rate of change and the instantaneous rate of change are concepts that take into account the ratio of the change of function value to the change of domain, the multiplicative rate of change is a concept in which the ratio of the function values is taken into consideration.

$$\frac{f(x + \Delta x)}{f(x)} = C_{\Delta x} \text{ (} C_{\Delta x} \text{ is a constant defined when } \Delta x \text{ is determined.)}$$

Confrey & Smith (1994) discussed the 'units' students perceived based on their perception of the rate of change in the function as additive and multiplicative rate of change. Especially, they saw that adjusting the multiplicative expression of the rate of change could make the concept of change rate more robust, and that it should be focused on the mental structure of the unit rather than the unit analysis including the standard unit. Ellis (2011) suggested the approach of rate of change as an alternative approach to understanding exponential growth and emphasized the importance of covariance and continuous variables.

In relation to this study, it is necessary to examine the study of Lee, Yang, & Shin (2017). In Lee (2017a), students constructed the speed-time function for distance-time function $y = \frac{x^3}{3}$ by constituting the average speed from time a to k as $\frac{\frac{k^3}{3} - \frac{a^3}{3}}{k - a}$, and dividing the factor $(k-a)$ in the denominator and then substituting $a=k$. However, students experienced difficulties in dividing the factor $(k-a)$ in the denominator when the given distance-time function was exponential function $y=2^x$ (Lee & Kim, 2017). Lee et al. (2017) introduces the process of which the students who encountered such difficulties construct instantaneous rate of change at the moment of $x=0$ based on the understanding of the natural constant 'e'. In this paper, however, students did not construct the speed-time function of the distance-time function $y=2^x$. Lee & Kim (2017)'s study introduces a situation where a student using smooth reasoning tries various algebraic attempts to divide the factor $(k-a)$ in the denominator after constructing the average velocity $\frac{2^k - 2^a}{k - a}$ for distance-time function $y=2^x$ from time a to k . The result showed that the student using smooth reasoning made efforts to construct the expression for speed-time function representing the speed at that moment through the perception of instantaneous change, but in turn failed to construct the speed-time function for the distance-time function $y=2^x$.

This study will put together series of these discussions and first examine how students using chunky reasoning constitute speed-time function of distance time function $y = \frac{x^3}{3}$ through constructing expression for average speed, and then focus on how they construct the speed-time function which describes the change of distance-time function $y=2^x$.

III. METHODS

This study is a qualitative case study to deeply understand students' concept of average revealed in teaching experiments and to comprehend the implications associated with the situation. Teaching experiment is a research method to establish a sustainable model for activities in which learners construct mathematical concepts. Although the teaching experiment is not constrained by the existing teaching

method or curriculum, most of the situations presented to the learners are likely to be the existing curriculum because it refers to the previous research data as an important reference. In addition, it has strong experimental characteristics because the process of teaching experiment is not constituted according to the previously predicted plan, but in accordance with the student's response to the task.

In the teaching experiment, the first task is selected under the consultation between the researchers. From then on, the task is sequentially constructed through students' reaction to conversation or behavioral outcome. The situation in which the researcher presents a task in consideration of the student's reaction can be seen as the intentional 'setting' and the 'reactivity' involved in the researcher's influence on the research subject (Yang & Shin, 2014), but since the goal of qualitative case study is not to eliminate the influence of researchers but to understand and put use of it productively (Maxwell, 2012), it could be considered an appropriate method for case studies. In this study, after the end of each teaching experiment, 'On-progress analysis' is carried out and the next experiment is conducted by consultation between the researchers. In other words, the teaching experiment is carried out with the repetition of the next task according to the reaction of the subject and the consensus agreement process among the researchers (Glaserfeld, 1995). When the teaching experiment is finished after repeating the circulation process of teaching experiment progression→on-progress analysis→determining task for next session, the researcher conclusively makes a comprehensive analysis using data of the entire teaching experiment (student response record, conference log of researchers, video and transcript data of the teaching experiment). This process is called retrospective analysis. Through retrospective analysis the researcher finds meaningful implications related to the research topic.

1. Teaching experiment

The teaching experiment for this study was started with the aim of studying the concept of the ratio, and the experiment was carried out in a total of 20 sessions (about 70 minutes per session). Among the 20 sessions of teaching experiments, ones directly related to this study are 16, 17, 18, 19, and 20th session; 19th and 20th session deals with the constructing process of the students using chunky reasoning on the task of finding the speed-time function when the distance-time function is given as exponential function $f(x)=2^x$.

The experimental time each session for the teaching experiment was not set in advance; normally, the experiment was terminated when the researcher decided there was a need for consultation with other researchers to present the next task. Teaching experiment data were collected during the period from May 2016 to February 2017; the researchers and the three research students met in a separate space rather than in the classroom (Twice a week after school during the semester and around 9 a.m. every day during the break). The spatial condition for the teaching experiment was a place where a camera and an audio recorder are installed to record the responses of the research subjects, attached to a separate place where the research assistant teacher could observe the experiment while waiting.

The researcher in charge of the teaching experiment had 15 years of teaching profession; one research assistant teacher participated as an observer in order to improve and suggest direction to any errors made by the researcher in the experiment. The research assistant observed the teaching experiment through the dialogue between the researcher and the three research subject students while waiting in the staff's room outside the barrier where the teaching experiment was conducted. If the researcher saw the need for a discussion while conducting the experiment, the researcher would exit and discuss with the assistant researcher outside the barrier to receive help while the students were working on the task. The subjects did not recognize the existence of the observer because outside was not visible from the conference room where the teaching experiment was conducted.

At the end of each session, researchers, research assistants, and other co-researchers jointly analyzed the meanings of students' thoughts and behaviors as they watched recorded videos and student activity logs, and they designed their next teaching experiments based on mutual consensus. To put the agreed task into the actual teaching experiment, the researcher made the decision during the teaching experiment with the students. Likewise so, after all the researchers (researcher, research assistant teacher, and co-researcher) agreed on the design of teaching experiments for the next session, the researcher repeated the process of selecting the final task and conducting the next session of teaching experiment. The teaching experiment was carried out mainly through communication among students; when necessary, the researcher intervened but only at the level of sorting the opinion of a student to confirm to all the other students or making an inquiry to move on to the next level.

To help understand the context of the teaching experiment related to this study, a general flow of the 20 sessions of teaching experiment was tabulated as an appendix; roughly three groups of teaching experiment are related: 1) 5th and 6th session which is about the changes in perception and expression of the relation between distance-time function and speed-time function, 2) 10th session in which the students constructed a unique speed-time function on their own from cubic distance-time function, and 3) 16th, 17th, 18th, 19th, and 20th session which causes difficulties to constitute speed-time function with the same method because the given distance-time function is exponential. In particular, this study describes in detail the process of a chunky reasoning student constructing a speed-time function from an exponential distance-time function.

2. Characteristics of the research participants and tasks

The students who participated in the study had different ways of reasoning continuous change, but it was recognized through observation in the teaching experiments. The researcher did not deliberately select students with different reasoning methods for continuous change.

The subject students A, B and C are students the researcher watched for 2 months as a homeroom teacher; although the academic achievement in math was different, they were all active in expressing their

opinions and communicating with other students. The researchers confirmed the students' intention to participate and finalized the decision. For reference, the academic achievement levels of the students are quite diverse: student A is 1st degree level (top 4%), student B is 2nd degree level (top 4~11%), and student C is 4th degree level (top 23~40%) based on the grades evaluated by National Academic Achievement Assessment conducted in April 2016. This kind of intentional sampling has the advantage that researchers can get more information from teaching experiments (Lee, 2017a). The mathematical grades of the subjects are only for reference, and they are not related to the qualitative difference of the mathematical concepts formed by the students in the teaching experiments. In other words, no such evaluation of distinguishing the superior is made in the teaching experiment of this study.

The subject students are 10th grade students in general high school located in Seoul; they were confirmed to have no learning experience of limit and calculus before the first teaching experiment, but the prerequisite learning of math during the 8 months of teaching experiment could not be separately regulated. However, on the 19th and 20th session of the teaching experiment which is related to the subject of this research (Winter vacation in January, 2017), some students said to have heard the term limit and have tried to solve related problems in textbook, but they still said to not have learned the concept of calculus. Therefore, the researcher and other researchers agreed to make an analysis based on the assumption that the subject students did not have prior experience in calculus learning during the teaching experiment period.

The main task presented in the teaching experiment was to construct a speed-time function for the distance-time function $f(x)=2^x$. This assignment was constituted after the 10th session in which the students showed the procedure of dividing the factor in the denominator from the average speed formula when the given task was to construct speed-time function from a polynomial distance-time function; the researchers then agreed to constitute a task which 'cannot be solved by the corresponding procedural knowledge and deals with the change of function from the viewpoint of rate of change' and therefore came out with the task. The analysis of the 19th and 20th session which consists of the process of the teacher and the students' activities for the task will be explained in detail in IV. 'Study Results'.

3. Data collection and analysis methods

Out of 20 sessions of the teaching experiment, this study focuses on teaching experiment data of 19th and 20th sessions which are related to the discussion. In this study, a video camera was used to capture the mathematical activities of three students; in addition, separately recorded audio data was transcribed for analysis.

In addition, this study examines the pedagogical decision process and its aspects of transition by collecting activity logs composed by students, field notes prepared by the researchers, and conference log of researchers in constructing the next teaching experiment task; based on these data, the reason for the

revision and reconstruction that occurred during the teaching experiment is described altogether in IV. 'Study Results'.

IV. RESULTS

1. Focusing on student B's smooth reasoning, the process of constructing speed-time function for exponential distance-time function $y=2^x$

In order to construct a speed-time function for exponential distance-time function $f(x)=2^x$, student B showed a number of attempts to reduce the factor $(k-a)$ in the denominator of average speed formula $\frac{2^k - 2^a}{k - a}$. In particular, student B showed difference in constructing speed-time function from the distance-time function $f(x)=2^x$ when compared to the 10th session of the teaching experiment of which the distance-time function was cubic. When student B previously constructed the speed-time function from cubic distance-time function, the student showed attempts to explain the change by the object's motion; as for the distance-time function $f(x)=2^x$, by the procedural knowledge the student formed earlier, the task had seemed to be converted from 'finding the speed-time function from the distance-time function' to 'reducing $(k-a)$ in the denominator of the formula representing the average speed'

In effort to find the instantaneous velocity at time k for the distance-time function $y=2^k$, the students first obtained the formula for average speed from time a to k as $\frac{2^k - 2^a}{k - a}$ like the previous sessions (10th, 13th, 16th). Then they thought of the idea 'time k to k ' from the expression 'time a to k ' and considered it as the instantaneous speed at time k . In particular, student B tried to obtain the instantaneous speed at time k by substituting $a=k$ to the average speed formula; in doing so, the student tried to use the method of dividing factor $(k-a)$ in the denominator and then substituting $a=k$.

In other words, student B expressed difficulty in reducing the factor $(k-a)$ in the denominator of the average speed formula $\frac{2^k - 2^a}{k - a}$ algebraically, but didn't give up and showed effort trying in various ways. However, student B showed hesitation when the student acknowledged that the result came out differently if the newly found methods were applied to the distance-time functions handled earlier ($y=x^2$, $y=\frac{x^3}{3}$, $y=\sqrt{x}$). Various attempts of student B to reduce the factor are discussed in detail in Lee & Kim (2017); in particular, in finding value for $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$, series of processes in which student B's procedural knowledge is transformed into a stage for constructing new conceptual knowledge are well described.

Resultantly, student B failed to algebraically reduce the factor $(k-a)$ in the denominator of the average speed formula $\frac{2^k - 2^a}{k - a}$, and this process seemed to provide an opportunity for student B to reflect on his

procedural knowledge.

2. Focusing on student C's smooth reasoning, the process of constructing speed-time function for exponential distance-time function $y=2^x$

On the other hand, student C observed the entire constitutive procedure of student B and expressed that he himself will construct the speed-time function for distance-time function $f(x)=2^x$ in his own way different from student B's method. The method student C presented was 1) calculating the function value of $x=1,2,3, \dots$ for exponential function $f(x)=2^x$ and 2) plotting $(1, f(1)), (2, f(2)), (3, f(3)), \dots$ on the graph of $f(x)=2^x$, and then 3) to draw lines from 'the origin to $(1, f(1))$ ', ' $(1, f(1))$ to $(2, f(2))$ ', ' $(2, f(2))$ to $(3, f(3))$ ' sequentially and calculate the slope. Then he would use the result to construct the graph of the new function: 4) the slope of the line segment connecting the origin and $(1, f(1))$ would become the function value for interval $[0,1]$, the slope of the line segment connecting $(1, f(1))$ and $(2, f(2))$ would become the function value for interval $[1,2]$, and so on. Lastly, the graph can be deduced to its final form of the speed-time function for distance-time function $f(x)=2^x$ by 5) reducing the length of the intervals by half over and over again. Student C's contention can be seen to be due to his previous experience with constructing the speed-time function for the distance-time function $y=x^2$.

Following this procedure, student C constructed a step function graph showing the change of the distance-time function $f(x)=2^x$ when the width of the section was 1, as shown in Figure 4. From the diagram, '0~1' means time from 0 to 1 second and the expression $\frac{1-2}{-1}=1$ is an equation representing the

Figure 4 Student C's procedure of constructing the graph expressing the change of distance-time function $f(x)=2^x$ and the resulting step function graph

difference in distance divided by the time difference in the corresponding interval.

Particularly among the expressions in Figure 4, '0~1' means 'time 0 to 1' and student C did not separately distinguish the interval between $[0,1]$, $[0,1]$, and $[0,1]$. However, considering that both ends were marked when the graph was constituted, the researcher regarded the student's expression '0~1' as interval $[0,1]$ and proceeded with the discussion.

However, after student C constituted the graph for interval length of 1, the student expressed grave difficulties in calculating the values while constituting the case for the interval length of $\frac{1}{2}$. For example, when constructing a new function that represents the change of distance-time function while the length of the interval is $\frac{1}{2}$, the student needed to calculate the function value of $\frac{2^{\frac{7}{2}} - 2^4}{\frac{7}{2} - 4}$ for the interval $[\frac{7}{2}, 4]$; values such as ' $\frac{7}{2}$ ' caused trouble in approximating and computing even with a calculator, and the student also had trouble expressing it on the graph. The bigger problem was that student C had to repeat this calculation procedure and at the same time, deal with the situation in which the length of the interval is continuously decreasing to $\frac{1}{4}$ or $\frac{1}{8}$ and so on. Student C later acknowledged this and seemed to be burdened. In this process, student A pointed out that his method was somewhat similar to that of student C, but he said he was simply observing the formula without calculation for possible patterns; the researcher agreed with this opinion and advised student C to do the calculations later and proceed with the steps with formulas.

Hence, student C summarized the calculation results as shown in Figure 5 to construct a graph of the new function that shows the change of the distance-time function $f(x)=2^x$ when the length of the interval was $\frac{1}{2}$. From the diagram, '0~0.5' means time from 0 to 0.5 second and ' $-2 + 2\sqrt{2}$ ' next to it is the

Figure 5 A picture depicting the calculations for constructing a graph representing the change of distance-time function $f(x)=2^x$ when the length of the interval is $\frac{1}{2}$

0 ~ 0.5	$-2 + 2\sqrt{2}$
0.5 ~ 1	$-2\sqrt{2} + 4$
1 ~ 1.5	$-4 + 4\sqrt{2}$
1.5 ~ 2	$-4\sqrt{2} + 8$
2 ~ 2.5	$-8 + 8\sqrt{2}$

average speed at that interval $\left(= \frac{2^1 - 2^{0.5}}{1 - 0.5} \right)$.

Student C then arranged the results of both endpoints for interval length 1 and $\frac{1}{2}$ as shown on Figure 6 and tried to observe the change.

Particularly in Figure 6, the red mark indicates the fourth term of the sequence, which the researcher could imply that student C tried to find a pattern from the table comprising of arrangement of the endpoints. Moreover, from the fact that student C wrote the results for interval length of $\frac{1}{2}$ under the results for interval length 1, one could imply that the pattern student C tried to find was not only the sequence for the fixed interval length of 1, but also the sequential pattern as the interval was repeatedly reduced to half.

Nevertheless, student C could not form an adequate rule from the arranged results shown on Figure 6; student C again constructed a table showing the change of distance-time function $f(x)=2^x$ when the length of the interval is $\frac{1}{2}$ as shown in Table 3 and obtained values of average speed in the small interval. This

action of student C seemed to be an endeavor to try making new attempts to find some sort of pattern rather than an intentional activity for a specific purpose.

Student C obtained the average speed from $\frac{1}{2}$ second to 1 second in Table 3 by $\frac{2^{\frac{2}{2}} - 2^{\frac{1}{2}}}{\frac{2}{2} - \frac{1}{2}}$ and organized the formula as $2 \left(2^{\frac{2}{2}} - 2^{\frac{1}{2}} \right)$. The student then repeated the procedure to each interval and obtained the orderly results of $2 \left(2^{\frac{1}{2}} - 1 \right), 2 \left(2^{\frac{2}{2}} - 2^{\frac{1}{2}} \right), 2 \left(2^{\frac{3}{2}} - 2^{\frac{2}{2}} \right), \dots$

Figure 6 A picture showing values of both endpoints student C arranged from results for interval length of 1 and $\frac{1}{2}$

Table 3 Table showing the change of distance-time function $f(x)=2^x$ for the interval length $\frac{1}{2}$

x	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$
$f(x)$	$\frac{1}{2^2}$	$\frac{2}{2^2}$	$\frac{3}{2^2}$	$\frac{4}{2^2}$

Figure 7 A figure showing the generalized formula for average speed in small interval

$$\int (2^{\frac{1}{2}} - 1) 2^x \} \times 2$$

Figure 8 An illustration showing the activity of student C in sequence when the width of the section is $\frac{1}{4}$

At the same time, while obtaining these results, student C tried out writing $2^{\frac{2}{2}} = 2^{\frac{1}{2}} 2^{\frac{1}{2}}$ and then wrote $2^{\frac{3}{2}} = 2^{\frac{2}{2}} 2^{\frac{1}{2}}$; the student then applied this to the former result, changing the expression into $2(2^{\frac{1}{2}} - 1)$, $2 \times 2^{\frac{1}{2}}(2^{\frac{1}{2}} - 1)$, $2 \times 2^{\frac{2}{2}}(2^{\frac{1}{2}} - 1)$, $2 \times 2^{\frac{3}{2}}(2^{\frac{1}{2}} - 1)$, ...

After constructing these rules, student C made a formula as shown on Figure 7.

Student C applied these activities to case in which the length of the interval is $\frac{1}{4}$. That is, as in the case for interval length $\frac{1}{2}$, student C drew a correspondence table of $y=2^x$ when the length of the interval is $\frac{1}{4}$, calculated the average speed for each interval, and organized the results to obtain generalized functional

expression. Figure 8 shows the activity of student C in sequence when the length of the interval is $\frac{1}{4}$.

Finally, Student C constructed a general relation that shows the change of average speed when the width of the section is $\frac{1}{n}$, as shown in Figure 9.

At this moment, student B, who had been watching the final result of student C, expressed that he had seen the relation formula of student C before and went on modifying $n(2^{\frac{1}{n}} - 1)$ to $\frac{(2^{\frac{1}{n}} - 1)}{\frac{1}{n}}$ and stating that becomes $\ln 2 (= \log_e 2)$ as n continues to increase.

Student C also expressed that he understood what student B was saying since student C had previous experience of watching student B's process of forming instantaneous speed at $x=0$ for the distance-time function $f(x)=2^x$. Student C further agreed on expressing the final result of the new function showing the change of distance-time function $y=2^x$ as $\log_e 2 \times 2^x$. Using these results, the students could explain that the value $\log_e 2 \times 2^k$ in $x=k$ can be described as the instantaneous speed at $x=k$ in the motion of the object following the distance-time function $y=2^x$.

Meanwhile, student A told that there is no original way of his own in expressing the change of distance-time function $f(x)=2^x$, and often used the expression that he said "I did not know what to do.". Thus the teacher advised to choose either way of the other students that seems more preferable and continue on with it if there is nothing much to do otherwise; student A followed student C's method and drew results in respect to distance-time function $f(x)=2^x$ for the interval lengths 1, $\frac{1}{2}$, and $\frac{1}{4}$ on the same coordinate plane as shown on Figure 10.

Also, as well for the process of calculating the average speed for each interval in regard to the change of the interval length, student A followed student C's method and expressed the organized results as Figure 11.

Student A differed from student C's way of expressing the length of interval as $\frac{1}{n}$ and used $\left(\frac{1}{2}\right)^n$ for the situation in which the length of interval halved as 1, $\frac{1}{2}$, $\frac{1}{4}$... but the structure of the procedure and result

Figure 9 General relation showing the change of average speed when the width of section is $\frac{1}{n}$

The image shows a handwritten mathematical formula: $(\text{간격이 } \frac{1}{n} \text{ 일 때}) = n(2^{\frac{1}{n}} - 1)2^x$. A blue line points from the text "(간격이 1/n 일 때)" to a blue-bordered box containing the text "As interval is 1/n".

Figure 10 Drawing of the results in respect to distance-time function $f(x)=2^x$ for the interval lengths 1, $\frac{1}{2}$, and $\frac{1}{4}$ on the same coordinate plane

Figure 11 Formula representing the change of average speed for different intervals in distance-time function $f(x)=2^x$

$$y = 2^n \left((2^{\frac{1}{2}n} - 1) 2^x \right)$$

was very similar to that of student C. Also, student A agreed as well to student B's modification to the student C's result and finally stating the result as $\log_e 2 \times 2^x$.

V. DISCUSSION AND CONCLUSION

1. Method by student C, who uses chunky reasoning, to constitute the speed-time function from the exponential distance-time function $y=2^x$ and its difference to the other students with different reasoning for continuous change

In constructing speed-time function for exponential distance-time function $y=2^x$, relevant distinction was observed between student B who used smooth reasoning and student C who used chunky reasoning, and this study is depicted from student C's point of view. Student B using smooth reasoning first derived the formula for average speed from time a to k . Then the student used the term 'average speed from time k to k' ' and was able to express the term instantaneous speed at time k . Whereas student C using chunky

reasoning had trouble accepting the method of student B constituting instantaneous speed from distance-time function.

However, student C using chunky reasoning was also able to explain the change of exponential distance-time function $y=2^x$ in a separate way. While doing so, the student constituted and presented a new formula, referring to it as a speed-time function which expresses the change of distance-time function.

In order to explain the change of distance-time function $y=2^x$, Student C divided the time domain (x-axis) to small, equidistant intervals and calculated the average speed for every interval. Then the student used them as function values for the intervals to draw a new graph in step form (as shown on *Figure 10*) and referred it to a speed-time function expressing the change of distance-time function. Student C then tried to observe the graph's change by dividing the time domain into smaller intervals. This, which we presume, is due to the previous experience of constituting a speed-time function in the same way for quadratic distance-time function.

When difficulty occurred in observation of graph's change by downsizing intervals as for exponential distance-time function, student C chose to observe the change in average speed of the intervals by altering the initial points of each interval. In such way, student C was able to constitute the algebraic formula shown on *Figure 11*, which the student then explained that speed-time function expressing the change of distance-time function could be constituted with infinitely large n .

Seemingly, the speed-time function constituted by student C does not differ from student B's in the algebraic point of view. However, following two distinctions differ from speed-time function student B constituted using instantaneous speed at time k .

First of all, the speed-time function student C constituted can be considered as the final form of step function graph displaying average speed on intervals. Though student C examines the change by dividing the graph into smaller sections, the fact that the student 'presented the resulting change of distance-time function as a whole graph and not as a specific section of the step-form function' indicates that the student is examining the change of the distance-time function in a comprehensive viewpoint. This shows a comparison with student B, who examined the change of distance-time graph in particular section first and then to a single point which corresponded to the algebraic formula as whole.

Another notable feature is that, student C still confessed trouble when confronted with student B's concept for understanding instant speed. Student C used chunky reasoning when understanding continuous change. In other words, the student could infer changes very close to an instant by dividing chunks into smaller parts repeatedly, but the student often had trouble expressing the change when it came to speed at the exact moment. Student B who used smooth reasoning, on the other hand, constituted instant speed to argue average speed. This contrast shows that even with apparent same result, interpretation or application of the output could be different. To sum up, both students using smooth reasoning and chunky reasoning could all reach the derived function exhibiting the change, but the two differed in ways to approach the concept, and they showed how there could be a contrast when

interpreting function values of the derivative.

This provides an important implication in school fields if the main focus is not on the practice of application but on the very concept of differentiation itself. Whether the base of reasoning (smooth reasoning and chunky reasoning) has an effect on learning the concept of differentiation is yet a matter to discuss with further studies. Nevertheless, this research suggests that there is an experimental basis to hypothesize that reasoning of continuous change can affect the process of constructing the concept of differentiation, and it is indeed notable. Presuming that current differential math education in school does not consider the difference in ways of reasoning toward continuous change, though this research doesn't necessarily concern 'how to teach the must-taught differentiation well', but has its significance concerning what would be the 'natural way for students to construct the idea of differentiation.'

2. Educational implications

This research is a case study that analyzes the process of constructing the speed-time function of the distance function $f(x)=2^x$ by students who have different reasoning methods for continuous change. The focus in this study about students' methods for construction is distinguished from learning the derivative of the exponential function in school mathematics. Learning the derivative of the exponential function in school mathematics is done according to the teacher's purpose and method which depends on the learning objective; whereas in this study, the reason for taking interest in the process of constructing the speed-time function of the distance function is to analyze 'student's process of constructing the speed-time function for exponential distance-time function' in order to understand students' unique comprehension of mathematic and gain insight to 'students' methods for reasoning continuous change' through the procedure.

Particularly in this study, after student B who infers through smooth reasoning constructed procedural knowledge of constructing speed-time function from a distance-time function, student B displayed a reflective approach to prior procedural knowledge when the student encountered exponential distance-time function which could not be solved by the previously constructed procedural knowledge. Also, student C, who used chunky reasoning in a way of sectional rate of change to infer continuous change, constructed the speed-time function for exponential distance-time function by dividing the interval into smaller sections, expressing the average speed for each interval, and reducing the length of the interval. This is not to argue that any reasoning method is more appropriate for obtaining the derivative of exponential function, but rather to present experimental evidence that difference in reasoning methods for the continuous change can lead to difference in mathematical results.

On the other hand, this work is similar to Lee et al. (2016), Lee (2017a), and Lee & Kim (2017) in that three students with different reasoning methods for continuous change have analyzed the process of constructing speed-time functions from a distance-time function. However, there is a distinctive part of

this study that distinguishes it from these studies.

First of all, studies by Lee et al. (2016), Lee (2017a), and Lee & Kim (2017) show three students constructing different mathematical results depending on different reasoning methods for continuous change. However, these courses showed smooth reasoning students leading the construction process, whereas chunky reasoning students could understand smooth reasoning students' process but tried to construct results with their own method of reasoning.

Particularly, in Lee & Kim (2017), in a special situation of $x=0$, student B who uses smooth reasoning took the lead in constructing mathematical results for instantaneous change of distance-time function $y=2^x$. However, it did not cover the process of constructing the speed-time function that could explain the overall change of distance-time function $y=2^x$.

On the other hand, this study introduces the process of constructing a speed-time function which can explain the overall change of the distance function $y=2^x$, led by student C who uses chunky reasoning. In particular, it is meaningful in that the result of student C is mathematically identical to the exponential function $y=2^x$ introduced in school mathematics, but was revealed through the construction process of student C.

If the research on students' reasoning and expression for change in exponential function becomes more of interest in domestic and foreign countries, the necessity for accumulation of experimental research and case studies on students will further be required. This study shows its significance in terms of empirically revealing how students structure the change of exponential function, based on discussions of students' perception of the relationship between continuously changing variables of time, speed, and distance and discussions on reasoning methods in perceiving continuous change.

However, we believe that continuous verification of the results presented in this study is necessary; it is also necessary to conduct a reflective study on the implications of this study through the follow-up studies on where the difference of understanding method of students' continuous change begins and how this difference is related to concept formation of limit.

In particular, this study developed through an observation in the 20th teaching experiment that the subjects' reasoning for continuous change is different. Therefore, based on the results of this study, a follow-up study is also needed to examine how each of the students with different reasoning methods for continuous change constructs the speed-time function from the distance-time function.

CONFLICTS OF INTEREST

No potential conflict of interest relevant to this article was reported.

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