Original Article

Development of Socio-Critically Open-Ended Problems for Critical Mathematical Literacy: A Japanese Case

Yuichiro Hattori^{1†}, Hiroto Fukuda², Takuya Baba³

¹Associate Professor, Kochi University, ²Lecturer, Okayama University of Science, ³Professor, Hiroshima University, Japan

Abstract

The purpose of this study is to propose *socio-critically open-ended problems* (SCOEPs) as a novel theoretical framework for nurturing students' critical mathematical literacy while respecting diverse values embedded in trans-scientific problems. First, we outline the socially open-ended problem—which is of current interest to Japanese researchers of critical mathematics education—and describe its nature and significance. Second, we derive issues from current research on social justice and ethics in mathematics education using a literature interpretive methodology and build a theoretical framework of SCOEPs to develop socially open-ended problems. We present several potential examples of classroom practices based on the SCOEPs framework that were implemented in Japanese schools to explore the impact of these questions on student's engagement and thinking processes. We found evidence that the objectives of nurturing both social judgment skills within an ethical framework, as well as fostering mathematically and socially diverse solutions for authentic problems are integrated in SCOEPs. The framework can be described as the coexistence of the process of fostering social decision-making through mathematical thinking and the process of critically considering mathematical thinking to achieve social justice. This proposal has significant implications for the future directions of mathematics education in the 21st century.

Keywords: critical mathematical literacy, ethics, socially open-ended problems, socio-critically open-ended problems, values

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†Correspondence: Yuichiro Hattori, hattori@kochi-u.ac.jp

ORCID: <https://orcid.org/0000-0003-0501-1983>

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I. PROBLEM STATEMENT AND RESEARCH AIM

Trans-scientific problems of which contemporary society abounds are ‘questions which can be asked of science and yet which cannot be answered by science’ (Weinberg, 1972, p. 209). The COVID-19 pandemic, as a contemporary example, has presented the international community with an unprecedented global crisis. Although the number of infected people can be simulated scientifically and technologically, based on certain mathematical models, the timing to invoke a state of emergency is a social decision that can never be solved by purely mathematical means. If the purpose of mathematics education is to develop citizens in society, an important question to ask is whether current mathematics education truly nurtures the skill to make social and mathematical decisions while accommodating mutual interests and trade-offs.

Comprehensive decision-making skills—based on both public and personal values—are required of citizens, as society shifts away from seeking one correct answer towards improved approaches and answers from multiple options. It is necessary to reconsider the current role that mathematics education plays. As Ernest (2012) suggested, the first philosophy in mathematics education is ethics. Thus, research into the ethical aspects of mathematics education has been promoted in recent years (e.g. Atweh & Brady, 2009; Skovsmose, 2019). The concept of critical mathematics education proposed by Skovsmose (1994) suggests the necessity for a deeper understanding of social situations and thus for critical citizenship through mathematics, which is consistent with an approach to dealing with trans-scientific problems. The research on critical mathematics education, however, has been positioned as an *avant-garde approach* to mathematics education (Ernest, 2010), while the absence of a coherent curriculum is a noted issue (Brantlinger, 2013). This research contributes to a paradigm shift in mathematics education research in the 21st century, by constructing a model curriculum for critical mathematics education in Japan that harmonises this approach with Japanese teaching culture. Mathematics education in Japan has been internationally appreciated as having a problem-solving approach that emphasises the structural aspects of mathematics (Stigler & Hiebert, 1999; Shimizu, 2009). We argue that it is a daunting but important task to consider the possibility that critical mathematical literacy—the social and mathematical skills—can be nurtured in a context that emphasises explanations and decision-making based on ethics and values.

Bearing this socially complex context in mind, the purpose of this study is to propose socio-critically open-ended problems (SCOEPs) as a new theoretical framework for nurturing students’ critical mathematical literacy while respecting diverse values in solving trans-scientific problems.

II. RESEARCH DESIGN

Our research method first reviews the literature on socially open-ended questions (Baba & Shimada, 2019), which has garnered the attention of current Japanese researchers of critical mathematics education. Second, we interpret the literature on social justice and ethics in mathematics education to approach

trans-scientific problems. We use these insights to develop a theoretical framework for socio-critical open-ended problems, which is an extension of socially open-ended problems, with more emphasis on the perspective of social justice and ethics. This theoretical framework can be used to support the philosophy of critical mathematics education at the curriculum level.

III. SOCIALLY OPEN-ENDED PROBLEMS AND THEIR OUTCOMES

1. Critical mathematics education and socially open-ended problems

Critical mathematics education recognises mathematics as a tool for identifying or analysing important features of society (Skovsmose & Nielsen, 1996). The teaching and learning process should seek to provide students with the opportunity to develop critical capabilities as necessary qualities for their participation in the further democratisation of society (Skovsmose, 1994). These aspects of mathematics are consistent with global developments and the spread of fundamental skills, as represented by OECD-PISA, key competencies, and 21st-century skills, among others. Skovsmose (1994) describes the objective of critical mathematics education as:

...not to concretise mathematics but to see in what way mathematics could develop from a broad context requiring the use of mathematics. If mathematics is everywhere in daily life situations, then it need not be necessary to develop any artificial concretisation. Instead, the prerequisite must be to create open-ended situations and in these, to let mathematics grow (p. 79).

In critical mathematics education, mathematics is used to manage the phenomena of society (Skovsmose, 1994). The learner recognises and identifies the role of mathematics in society. Critical mathematics education aims to cultivate critical citizenship, with an emphasis on the social context. It is consistent with the recent concept of mathematical literacy emphasising mathematical skills that are required for citizens to shape society (Jablonka, 2015).

In Japanese mathematics education, the development of mathematical thinking based on problem-solving has been consistently important and fostering mathematical thinking has been prioritised as a major goal of mathematics in Japanese schools (Hino, 2007). In contrast to mathematically open-ended problems, socially open-ended problems (Baba & Shimada, 2019) have recently attracted much attention. Indeed, theoretical and empirical research based on socially open-ended problems are being conducted in Japan and abroad (e.g. González & Chitmun, 2017; González et al., 2018; Naito & Hino, 2018; Nakawa, 2019). The term ‘mathematically open-ended problems’ is coined to show the difference. A socially open-ended problem offers ‘mathematically and socially diverse solutions aimed at nurturing social judgment

through mathematical thinking' (Baba, 2009, p. 52), and is in harmony with the philosophy of critical mathematics education. Thus, the socially open-ended problem is an extension of the open-ended approach (Becker & Shimada, 1997) that has been employed in Japan to develop the above mathematical thinking. As noted by Borasi (1990) and Silver (1993), values are expressed in the process of problem-solving. Iida et al. (1994) also clarified that values are expressed in the process of studying mathematically open-ended problems. Through studying and extending these, Baba (2009) formulated socially open-ended problem-solving as a method for mathematics education in the context of globalisation, with its diverse coexisting values.

Socially open-ended problems are characterised by a recognition of the social contexts of a problem. These contexts include complex and authentic issues that are occurring or may occur in everyday life, and which interest all the citizens in society (Shimada & Baba, 2014). Solving these problems involves values. Social values in this research are defined as 'children's beliefs and thoughts on mathematical problem-solving in the situation related to social life' (Shimada & Baba, 2013a, p. 84). Social value relates to interactions with other people, groups, and society, whereas personal values relate to the individual. The demarcation between social and personal values is very subtle. Socially open-ended problems have the following characteristics. They recognise (i) the emphasis on social context, (ii) the authenticity of problems, (iii) interpretation of condition, (iv) social values inherent in problem-solving, and (v) mathematical models¹⁾ (Shimada & Baba, 2014). An authentic problem is a 'problem given to children along with a concrete theme, goal, situation, not with a universal description' (Shimada & Baba, 2014, p. 77). A socially open-ended problem is developed to allow students to take into account diverse conditions in creating diverse solutions (Shimada & Baba, 2014). A comparison of mathematically and socially open-ended problems is presented in *Table 1*.

Table 1 indicates that a mathematically open-ended problem is a traditional one, characterised by

Table 1 Comparison of two types of open-ended problems (Baba, 2007, p. 22)

	Mathematically open-ended problem	Socially open-ended problem
Objective	To nurture mathematical thinking	To nurture mathematical thinking and judgement based on mathematical thinking and associated social values
Problem	To allow mathematically diverse solutions	To allow mathematically diverse solutions and associated social values
Method	Discussion on mathematically diverse solutions and their generalisation and symbolisation	Discussion on mathematically diverse solutions and the associated social values

1) Shimada (2017) defines a mathematical model as 'a model that represents an event according to a certain purpose using mathematical representations such as numbers, equations, diagrams, tables, and graphs that can be processed mathematically' (p. 110) to promote practical research on socially open-ended problems. We use this definition of a mathematical model for our discussion. Mathematical modelling is a series of processes that starts from real-world problems, translates them into the mathematical world, processes them, and returns them to the real world (Shimada & Baba, 2014).

and promoting mathematical thinking, while the mathematical thinking demonstrated in socially open-ended problems is regarded as a method of problem-solving. Moreover, socially open-ended problems are characterised by their active demonstration of the social values of students in solving them. These social values, exercised by students, are evident in problem-solving because the problems provided are, to some extent, social in nature.

2. Specific examples of socially open-ended problems

Let us consider the problem shown in *Figure 1*.

Certainly, it is easy to provide a model answer based on the remainder in the answer to a division problem, whereby a total of 8 couches are required since $30 \div 4 = 7$ remainder 2. However, the answer cannot be fixed when we consider real social concerns. For example, the solution of 4 people sitting in 6 couches and 3 people in 2 couches—despite being an equivalent solution to 8 couches—could be an alternative option in which the average concept is used as a mathematical idea. A solution of 10 couches, whereby all students get along well and are seated in groups of 3, would also be a possible mathematical solution based on the children's related social value of fairness (the mathematical formulation here is $3 \times 10 = 30$). The mathematically open-ended approach focuses primarily on the mathematical aspects of

Figure 1 Socially open-ended problem 'Number of couches required'

There are 30 children. How many couches, each seating up to four people, should be available?

Figure 2 Socially open-ended problem 'Matoate' at the elementary school level (Baba & Shimada, 2019, p. 176)

At a school cultural festival, your class offers a game of hitting a target with three balls. If the total score is more than 13 points, you can choose three favorite gifts. If you score 10 to 12 points, you get two prizes, and if you score 3 to 9 points, you get only one prize. A first grader threw a ball three times and hit the target in the 5-point area, the 3-point area, and on the border between the 3-point and 1-point areas. How do you give the score to the student?

the problem, while the socially open-ended problem approach utilises both the mathematical solutions and the associated social values to foster decision-making ability (Baba & Shimada, 2019). In recent years, several educational practices based on this teaching method have been devised, including:

The socially open-ended problem (Matoate) in *Figure 2* was introduced by Baba and Shimada (2019) in the elementary school by which stage mathematical models based on social values were common in the classroom, including kindness to the first-grader and fairness and equality. It is apparent that, when comparing the beginning and the end of the lesson, a refinement of students' thinking about mathematical models occurs. Regarding social values, a shift is evident. About one-third (31.6%) of the students were influenced by other students' mathematical models and social values, indicating that they amended their social values during the lesson. This Matoate problem has also been implemented in Bangkok, Thailand for 12th graders in public high schools (González & Chitmun, 2017). In Tanaka and Hattori's (2020) socially open-ended problem 'Purchasing a mobile phone' (*Figure 3*), junior high school students used mathematics to make decisions, based on their values, when deciding which mobile phone plan to recommend to their teacher. In the study, students who built mathematical models and tried to persuade others to submit alternative proposals based on these models as well as students who deliberated and sought to further strengthen their own opinions from other evidence while considering others' opinions were identified. In this lesson, the students also reviewed their ideas carefully in response to the arguments of their fellows or revised their decisions when convinced by their classmates. This demonstrates how students make a variety of decisions beyond mathematical judgment in discussions, influenced by social values such as economic efficiency and preference.

In addition to this, the development of socially open-ended problems based on critical mathematics education was promoted centrally in Japan by Baba (2016, 2017, 2019, 2020), the third author of this paper. The research theme—'Fostering Critical Thinking in Mathematics Education Based on Critical Mathematics Education'—was selected as one of the open research projects by the Japanese Society of Mathematics Education in 2016, 2017, 2019, and 2020. In this project, Shimada (2020), one of the members of the research group, classified the categories of social open-ended problems into four: 'distribution',

Figure 3 Socially open-ended problem 'Purchasing a mobile phone' at the secondary school level (Tanaka & Hattori, 2020, p. 3)

Our teacher is thinking of purchasing a new mobile phone. He would like to purchase it from one of the following three companies.			
Plan \ Companies	A	B	C
Amount of data that can be used monthly and fees	25 GB for JP¥9000	12 GB for JP¥5000	6 GB for JP¥3000
Calling charges	Free	JP¥50 per minute	JP¥150 per minute*
*The upper limit of call charges of Company C is JP¥9000, with no additional call charge beyond this.			
The monthly charge of each company is calculated as: Monthly charges=Monthly data usage charge+Call charges. In this case, which company's plan would be the most economical for our teacher?			

‘rulemaking’, ‘selection’, and ‘planning and prediction’, and organised them as shown in *Table 2*.

In Nakawa and Kosaka (2019), a research group proposed a socially open-ended problem, ‘distribution of oranges’, for kindergarteners. This problem asked the students to think about how to divide nine oranges among everyone (6 students). Based on *Table 2*, this practice falls into the ‘distribution’ category. The ‘Number of couches required’ in *Figure 1* is also in the ‘Distribution’ category, the ‘Matoate’ in *Figure 2* is in the ‘Rulemaking’ category, and the ‘Purchasing a mobile phone’ in *Figure 3* would be in the category of ‘Selection’.

3. Developing socially open-ended problems

Research on socially open-ended problems is important in manifesting children’s social values that, to date, have been ignored in mathematics education (Iida et al., 1994). Mathematical models based on children’s social values are built for social context-based problems, and various solutions are submitted in the classroom through discussion with others. However, if critical mathematics education intends to provide values-based critique, it is necessary to clarify the presupposed or base values when commencing the critique (Ernest, 2010). The values under focus are general educational values (Bishop et al., 1999), which include embedded ethical values. Boylan (2016) highlights two types of ethics: *universal* and *situational*. He also proposes four dimensions, indicating different ethical responsibilities, when examining ethics: relationships to others; social and cultural relationships; ecological; and the relationship to self. ‘Ethical action is always provisional. The best we can do is move step-by-step, and as we do this our actions change the world’ (p. 407). To navigate ethics in situations like this, it is important to discuss the situation in relation to ethical constructivism (Skovsmose, 2018), including the situational and pluralistic

Table 2 Categories of socially open-ended problems (Shimada, 2020, p. 146)

Categories	Aims
Distribution	To critically consider various values and mathematical models of fairness and compassion through the distribution of finite objects and people. We will also critically consider the intentions behind the distribution when it has already been achieved through policies.
Rulemaking	To critically consider various values and mathematical models such as fairness and compassion through rulemaking (system making). In addition, when rules (systems) have already been created by others (policies), students will critically consider the intentions behind their creation.
Selection	To critically consider a variety of values and mathematical models through the choice of goods (cars, mobile phones, candies, etc.), people (athletes, elections), occupations, and schools. We will also critically consider the intentions of the choices that have already been made by others (policies).
Planning and prediction	To critically consider various values and mathematical models by making plans and predictions for the near and distant future, utilising temporal, spatial and economic factors. Students will also critically consider the intentions behind the plans and predictions when they have already been made by others (policies).

In the table, Shimada (2020) also summarises the kinds of social values expressed by students in different social situations. In *Table 2*, only the categories and aims have been extracted.

nature of ethics in problem-solving—a major topic in mathematics education. Socially open-ended problems are considered a method to develop competence within a society that accepts diverse values to foster situational and pluralistic ethics in the classroom. There are two layers to the value perspective: problem-solving related values of individual students, and the values of the classroom that allow for the individuality of these values. Moreover, mathematical models constructed on these values are subject to debate depending on the situation. Mathematics is instrumental and essential for reasoning with its focus on finding means-to-ends and not underlying values (Ernest, 2010).

IV. CONTEMPORARY RESEARCH ON ETHICS IN MATHEMATICS EDUCATION

1. Response-ability

Atweh and Brady (2009) discuss responsibility as expressed by the term response-ability. Response-ability focuses on the values and talents of individuals. They identify a variety of prior mathematics education studies related to society and culture that challenge the dominant view of mathematics as a singular, objective, and value-free discipline that is isolated from human interest. They continue by outlining recent challenges from a social perspective and conclude by discussing the insufficient treatment of ethics (Atweh & Brady, 2009). Ultimately, they identify that the dominant Western academic tradition inadequately addresses ethics. At the root of the discussion is the principle that ‘Ethics relates to the face-to-face encounter with the other that precedes concepts and reflection’ (p. 274), emphasising the possibility of correspondence with society and others.

From this ethical perspective, in order for mathematics to contribute to the response-ability of the student as citizen, it should attempt to engage the student in meaningful and authentic ‘real world’ problems and activities that not only develop the mathematical capability but also develop an understanding of the social world and contribute to its transformation whenever possible (p. 274),

They thus show us the responsibility of mathematics education from an ethical perspective.

2. Authenticity of the context

Palm (2008) describes authenticity as follows:

... for a school task with an out-of-school task context to be authentic, it must represent some task situation in real life, and important aspects of that situation must be simulated to some reasonable degree. (p. 40)

Palm (2008) concedes that ‘a school task can, of course, never completely simulate an out-of-school task situation...’ (p. 40) and concerning the degree of authenticity of a simulation that ‘... under both circumstances, tasks may be developed that simulate more or fewer of the aspects with more or less fidelity’ (p. 40).

To develop word problems with authenticity, it is a prerequisite that the event described in the school task has taken place (or has a fair chance of taking place), and any questions posed are feasible in the real-life situation (Palm, 2008). Critically, what are the criteria for judging whether this lesson achieves authenticity? A given problem must originate within a real context while also reflecting ‘a problem of (the students’) personal affairs’ that should be solved. Where problematic contexts are provided to children, social responsibility is waived because it is an educational approach to simulate an authentic situation (Vos, 2018); thus, as an advantage of this educational approach, students are absolved from responsibility and can make mistakes. In Japan, for example, real multiple debt problems and socio-political issues tend to be skipped in mathematics lessons due to educational considerations. However, avoiding these problems that exist outside of school also denies the opportunity for students to use mathematics to address social issues (Weiland, 2017). Increasingly important research in recent years, incorporating social problems and social justice into mathematics education (Gutstein, 2003; Gutiérrez, 2013), methodically address classroom authenticity problems in fostering students’ ability to deal with trans-scientific problems. Teachers must ensure that learners understand questions about their society and use mathematics to interpret the world (Gutstein, 2003, 2006) to achieve a more ethical society and our future survival.

3. Socio-critical modelling

From an ethical viewpoint, mathematical modelling is a required characteristic of socially open-ended problems, as indicated by recent research on socio-critical modelling. Socio-critical modelling is based on the socio-critical approach in political sociology (Dede et al., in press) as the subjective sensing of activity through understanding the role of mathematics in society (Blum, 2015; Gibbs, 2019). Mathematical models are often used in societal decision-making, but the embedded assumptions or criteria of their construction and application are implicit, and generally express the worldviews of experts (Gibbs, 2019; Skovsmose, 1994). However, in the current era of trans-science (Weinberg, 1972) or post-truth (Matsushita, 2017), in addition to the worldviews of experts, the values of citizens towards society are indispensable elements. The criticism of the construction and application of mathematical models based on reflexive discussions with diverse values (Gibbs, 2019) is, thus, also part of the process in socio-critical modelling (Alrø & Skovsmose, 2002). Such socio-critical modelling has authenticity (Blum, 2015) and is consistent with the thematic approach practised in critical mathematics education (Skovsmose, 1994). Researchers agree that an emancipatory approach—a key concept of critical mathematics education—is related to socio-critical modelling (e.g. Kaiser & Sriraman, 2006). Skovsmose (1994) comments that an emancipatory approach

is ‘the result of a critique of ideology, and that means a freedom from stereotypes of thoughts’ (p. 19). Breaking from the stereotype that the decision-making of experts is always correct, modelling as critic (Barbosa, 2006, p. 294) encapsulates socio-critical modelling.

As Dede et al. (in press) suggest, general educational values—such as social justice, equity, social welfare, humanity, and altruism—are important in socio-critical modelling, and the method is a reflexive and inevitable discussion based on diverse values and ethics. In mathematics education research, the importance of values has been emphasised for decades (Bishop, 1991, 1999). Values in mathematics education are now regarded as the third wave, in parallel with the two major waves, of cognition and affection (Seah, 2010, 2018; Seah & Wong, 2012), including the publishing of a book focusing on values in mathematics education (Clarkson et al., 2019). In addition, the ethics underlying these values are positioned as the first philosophy in mathematics education (Ernest, 2012). More recently, Skovsmose (2020) identified sociology as an important addition to ethics and suggested that ‘the sociological dimension addresses the social formation of mathematics, while the ethical dimension addresses the mathematical formation of the social’ (p. 1).

V. THE PROPOSAL OF SOCIO-CRITICALLY OPEN-ENDED PROBLEMS

This research proposes SCOEPs in developing socially open-ended problems. First, as an objective of SCOEPs, the competency we aim to foster is critical mathematical literacy. This is a combination of ‘the ability to critically perceive mathematical thinking according to the situation’ and ‘the ability to make social decisions based on values using mathematical thinking as a method’, which can be achieved through discussing original socially open-ended problems. ‘Mathematics can make part of the readings and treatment of a crisis, which however could turn out to be misreadings and mishandlings’ (Skovsmose, 2019, p. 1). Those who solve problems must exercise caution in using mathematics, including assessing its applicability to certain situations in the context of an authentic problem involving social justice and ethics. Traditionally, mathematics lessons have endeavoured to achieve mathematical solutions. However, in SCOEPs, the decision not to use mathematics may be an applicable solution depending on the situation. SCOEPs are based on socially just and ethical values, searching for the best answer by discussing various problem-solving options. These include finding ways to answer a question in which mathematics is applied or to provide a social answer without venturing into a mathematical approach. Thus, the role of mathematics in society is highlighted more effectively through the development of value-focused mathematics lessons (Seah et al., 2016). *Table 3* compares the theoretical SCOEP framework to previous mathematical and socially open-ended problems. In general, problems that are designed to allow for multiple possible correct answers are called open-ended problems. Socially open-ended problems, as opposed to traditional mathematical open-ended problems, emphasise the richness of the social values that students express when solving problems. SCOEPs are a further development of the social aspect of

Table 3 *SCOEP Theoretical framework: Comparison of three open-ended problem approaches*

	Mathematically open-ended problem	Socially open-ended problem	Socio-critically open-ended problem
Objective	To nurture mathematical thinking	To nurture mathematical thinking and judgement based on mathematical thinking and associated social values	To nurture both the social judgment skill based on mathematical thinking and the skill to criticise the used mathematical thinking in a context to attain social justice and ethics
Problem	To allow mathematically diverse solutions	To allow mathematically diverse solutions and associated social values	To allow mathematically and socially diverse solutions for authentic problems involving social justice and ethics
Method	Discussion on mathematically diverse solutions and their generalisation and symbolisation	Discussion on mathematically diverse solutions and the associated social values	Discussion of mathematically and socially diverse solutions and values based on social justice and ethics

socially open-ended problems, in which the focus is more on solving any social problems that global citizens might face; the role of mathematics in social problems is also critically considered. The quality of the solution is also evaluated. Furthermore, the problem contexts in SCOEPs are more socially oriented than in socially open-ended problems; thus, the decision-making of learners is characterised by a greater emphasis on ethical aspects.

Since socially open-ended problems and SCOEPs involve constructing mathematical models and making decisions based on learners' social values, we expect learners to acquire both mathematical thinking and social judgment skills. Some might argue that this change in focus might result in the insufficient development of mathematical thinking. We are not denying the paradigm of mathematical open-ended problems. In fact, in Japan, several mathematical open-ended problems are included in elementary and junior high school textbooks to arouse students' interest and develop mathematical views and thinking (Hino, 2007). The study of mathematical open-ended problems should be promoted as an ineradicable task. It is noteworthy that in Japan, the percentage of students who think that mathematics is not useful in daily life is lower than the international average (Mullis et al., 2020). To achieve this task, it is necessary to consider the relationship between mathematics and society, instead of focusing solely on examinations (Nagasaki, 2010). Accordingly, we make a distinction between 'teaching mathematics' and 'teaching with mathematics'. This does not mean that we do not teach mathematical knowledge and skills, just that we should not only teach them. The coexistence of both is important. For example, in the study by Tanaka and Hattori (2020) — which aimed to foster students' critical thinking using socially open-ended problems — the critical thinking in mathematics education was differentiated into the critical thinking implemented while problem solving in the real world, and the critical thinking implemented while problem solving in the world of mathematics. In practice, we aimed to cultivate students who use mathematics to solve real-world problems. Considering the future of mathematics education, and the established goal of fostering mathematical thinking, critical consciousness and democratic civics through mathematics (Ernest, 1991)

should also be set as future goals of mathematics education.

In 2020, during the COVID-19 pandemic, an experimental lesson with the potential for SCOEP (Uegatani et al., 2020) was conducted involving Japanese senior high school students. This lesson was conducted in two 10th grade classes at a high school affiliated with a national university in Japan. In the national textbooks of Japanese high schools, the unit on probability is optional, and the concept of conditional probability is introduced in the unit. This lesson was planned as the conclusion to the study on probability. The students were asked to discuss the necessity, in their social lives, of requesting negative result certificates for COVID-19. The students were asked to read a newspaper article about professional sports teams and companies requiring players and employees to submit a negative test result certificate. The students were given the following fictional problem (*Figure 4*).

The students asked the teacher to provide the information that was missing in this scenario, which was provided as appropriate. The students were divided into several groups, each of which was assigned one of three roles—a male who supervised the part-time worker; a store manager; or a mayor. Each group was asked to discuss the validity of the negative test certificate from their perspective. The social problems associated with polymerase chain reaction (PCR) testing can be used as a highly authentic teaching tool for learning conditional probability. It has, however, been reported that even intelligent students in the experimental lesson intentionally ignored the results of probabilistic thinking and jumped to conclusive ideas, as evidenced in Student A's response presented in *Figure 5* (Uegatani et al., 2020, p. 48).

Figure 4 *The problem of the validity of submitting a negative test certificate*

[Question] A university student and part-time worker in a supermarket in a city, is requested by his manager to submit a negative test result certificate to continue to work. Is this request valid?

Figure 5 *Student A's opinion about the need for certificates*

It is inappropriate to consider if negative results of PCR tests should be submitted, based on mathematical probability. Certificates of negative results are prepared to doubly assure the results. While ordinary people judge whether the management of companies and city governments is right or wrong, many of them do not criticise such management by using various data at that time and calculating probabilities. We should consider and judge the same from the multiple perspectives cultivated in our daily lives.

Figure 6 *Student B's opinion about the need for certificates*

I do not know what we should actually do. If the precision of PCR tests is low, the value of certificates of negative results would be lost in the first place, but I think that this is a difficult issue as the precision is not low enough for us to affirm the same easily. After all, this issue will not be resolved till everything returns to normal.

Student A argued that mathematics had been used inappropriately and we need to consider non-mathematical factors. Student B expressed a different view as shown in Figure 6 (Uegatani et al., 2020, p. 48).

Thus, one of the characteristics of SCOEPs is the given social context in which students evaluate their use of mathematics. The pedagogical approach of this lesson was pre-designed to consider socially open-ended problems. However, the context of the given problem was authentic, involving ethical considerations, and the students considered the role of probability in socio-critical modelling. The experimental lesson allowed insufficient time for deep discussion as it was only an hour long. Consequently, this lesson should, in future, be further improved from the perspective of research design (Bakker, 2018).

Another potential example of SCOEP is the experimental lesson for second-year junior high school students in Japan displayed in Figure 7 (Fukuda, 2017; Hattori & Fukuda, 2019). Please refer to Fukuda (2020) for the detailed record. Students investigated the causes of global warming and countermeasures to prevent it by engaging in group activities in books and the Internet. This lesson in statistics education conducted a causal inquiry that required the students to summarise the discussion in a cause-and-effect

Figure 7 Cause and effect diagram and final decision making of Student C (Hattori & Fukuda, 2019, pp. 23-24; translated by the authors)

In my investigation of global warming, I found many possible causes. I knew that the emission of greenhouse gases, which is a worldwide problem, is one of the causes. Four of the six groups said that CO₂ was not the cause, but CO₂ is harmful to both humans and the earth even if there is really no connection. Therefore, I think that we need to reduce CO₂. In order to do so, we need to not waste our finite resources and take care of even small things such as saving electricity and water. I think that anyone can do something like that, so I will turn off the lights and water which I don't use.

diagram. Figure 7 and 8 show the diagrams reflecting the decisions made by two students as a result of their investigations.

The goal of the lesson is to foster critical citizenship (cf. Skovsmose, 1994) through statistics education from the standpoint of ethics for ‘meaningful and authentic “real world” problems’ (Atweh & Brady, 2009, p. 274). The students not only summarised the causality in cause-and-effect diagrams based on their investigations with objective data but also considered the actions they could take based on the results, which concerns response-ability. In addition, this lesson confronted a socially open-ended problem because it ‘attempts various approaches to the familiar social issue of “global warming” through causal inquiry using cause and effect diagrams as the method’ (Hattori & Fukuda, 2019, p. 23; translated by the authors). It is also a SCOEP in that the opinions of others and the cause-and-effect diagrams are objects of discussion.

The socially open-ended problem of the ‘number of couches required’ introduced in 3.2 can also be a SCOEP according to the framework. It considers the social values of fairness and the current social

Figure 8 Cause and effect diagram and final decision making of Student D (Hattori & Fukuda, 2019, p. 24; translated by the authors)

The causes of global warming are the increase in CO₂ and the influence of the sun. The causes of the increase in CO₂ are population growth and deforestation. However, we cannot do anything about population growth and the oceans. However, I think we can reduce deforestation, so we need to reduce it. The influence of the sun cannot be stopped directly. The acidification of the sea results in the release of CO₂, so I think it is better to reduce CO₂. Nowadays, electric cars and other electric devices are increasing, so I hope that this trend continues. To be honest, I don't know exactly how to prevent global warming, but I think that each of us should think about it seriously.

situation. Consequently, to prevent being infected by the coronavirus, we should be aware of social distancing, so the answer is that two children should be seated on each couch, resulting in the need for 15 couches. In SCOEPs, it is necessary to be conscious of society and to ponder on better solutions from various perspectives—not only mathematically.

In implementing SCOEPs in the mathematics curriculum, we should consider the balance between *top-down* and *bottom-up* processes for students (Ernest, 1991). In the top-down teaching-learning process, the goals of ethics and social justice are emphasised. Alternatively, the bottom-up teaching-learning process emphasises the starting point of ethics and social justice inherent in children's daily lives. Ethics are situational, and values vary across cultures. Each country has its cultural values, with different educational values in their mathematical modelling tasks (Dede et al., in press). Thus, different cultures have distinct goals and starting points for ethics and social justice, so the balance between top-down and bottom-up teaching and learning processes will also differ. A microcosm of society will be established in the classroom when SCOEPs are introduced into school education. Two processes coexist in the classroom: the teaching and learning process of mathematics through values—social formation of mathematics—and values through mathematics—mathematical formation of the social (Skovsmose, 2020). It can also be described as the coexistence of the process of fostering social decision-making through mathematical thinking and the process of critically considering mathematical thinking to achieve social justice.

Finally, we consider the following question: During the 21st century, another opportunity to reflect on mathematics education will arise from insights into the 2020 COVID-19 pandemic, such as the importance of trans-scientific problems. Even if there were no pandemic, how much would SCOEPs become a feature of the 21st century? This question can be approached from two perspectives: content-based and method-based sociality of SCOEPs.

First, we examine content-based sociality. There are four mathematical content categories (*Quantity, Uncertainty and Data, Change and Relationships, and Space and Shape*), and four contextual categories (*personal, occupational, societal, and scientific*) identified by PISA (OECD, 2018). Both category sets are related to the content of the problem—the first is the mathematical content, and the second is the context in which the problem is placed. The context is:

... the aspect of an individual's world in which the problems are placed. The choice of appropriate mathematical strategies and representations is often dependent on the context in which a problem arises, and by implication, there is the need to utilise knowledge of the real-world context in developing the model. (OECD, 2018, p. 29)

Traditionally, this context is ignored in solving a problem, and the validity of mathematics is restricted from going beyond the meaning of the content. Thus, the context, in this case, goes unrecognised. However, in the current PISA 2021:

Mathematical literacy is an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts, and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged, and reflective 21st-century citizens. (OECD, 2018, p. 7)

This has led to the need for 'each individual to know the role that mathematics plays in the world' within society, rather than simply being able to use mathematical skills or understand mathematical concepts. In this case, context is not something to be ignored but becomes necessary to foster the appropriate attitude of thinking within the context. For example, the Sustainable Development Goals (SDGs), as common goals of the international community for the 21st century, are targets set according to categories of problems in society—social problems are segmented. However, mathematics has its unique characteristics. To realise science that is made for society, it is important for problem-solving in mathematics to be associated with the context, thereby fostering the relationships between mathematics and society, mathematics and science, and between science and society.

Another perspective is the methodological sociality of SCOEPs. In recent years, PISA has highlighted that collaborative problem solving (OECD, 2017) plays an important role as a significant and necessary skill to be utilised in education and the workplace. Collaboration and communication skills, which are also regarded as 21st-century skills, are ways to attain knowledge, and the ability to collaborate with others is both the objective and the content that we need to master. Traditionally, as a tool for thinking and expressing ideas related to quantity and shape, mathematics education demonstrated its power in the formation of basic literacy and higher-order cognitive skills. In recent years, the construction of meaning in the classroom and relationships with others inside and outside the classroom have also been considered. For example, social construction refers to the construction of mathematical meanings through discussions and interactions with others, and instead of the construction of meanings (concepts) being set by an individual, it refers to the constructions made by the society of the classroom and outside the school. These concur with the method, purpose, and content discussed above. In SCOEPs, the social aspect may not be discussed, depending on how the context is managed. Therefore, not only the content but also the way to manage it—the method—plays an important role. The use of various models in the discussion can be linked to the development of collaborative problem-solving skills. Such classrooms show competing ethical commitments (Boylan, 2016), introducing a two-tiered value system including individual values and a classroom value system that accommodates them. The objective 'to nurture both the social judgment skill based on mathematical thinking and the skill to criticise the mathematical thinking used in a context to attain social justice and ethics' in SCOEPs and the method that 'allows mathematically and socially

diverse solutions for authentic problems involving social justice and ethics' are integrated. Thus, SCOEPs is a practicable and necessary proposal for mathematics education in the 21st century.

VI. CONCLUSION

In this research, we have built a theoretical framework for SCOEPs that extends socially open-ended problems (Baba & Shimada, 2019), to construct a model curriculum for critical mathematics education in engaging with trans-scientific problems. Based on this theoretical framework, it is initially implementable in the school mathematics curriculum in Japan. We believe that the SCOEPs we have proposed will be beneficial to all ages, from kindergarten through college²⁾. It will also be beneficial for teachers as well as students in terms of thinking about a better society. Making efforts to implement socio-political and ethical contexts into mathematics lessons leads to the uncovering of students' values. To date, teachers have avoided considering values as potentially disturbing in mathematics education and as unnecessary in understanding mathematical concepts (Iida et al., 1994). The concept of *post-truth* chosen as the Oxford Word of the Year 2016 expresses the idea that appealing to personal beliefs and emotions exerts a stronger influence on the formation of public opinion than objective facts (Matsushita, 2017). In applying post-truth to the academic context of mathematics education, both mathematical thinking and individual values should never be abstracted. The Japanese curricula (e.g. MEXT, 2018, 2008/2015) state that the teaching of mathematics needs to be related to moral education, although this has not yet been realised. A possible cause is the archaic view of mathematics education. We have reconsidered the role and meaning of mathematics education by presenting the rapidly developing novel approach to mathematics education as practicable teaching materials (SCOEPs) and educational practices. The significance of this study lies in the questioning of prevailing historical beliefs about mathematics education for the next generation. Against the backdrop of the COVID-19 pandemic, mathematics education must be carefully re-examined. Critical mathematical literacy is a skill that can be used to critique the claims of experts and make decisions on trans-scientific problems—through public discussions and collaboration among citizens who have diverse values—ensuring future survival. Finally, this research is significant in terms of the construction of a principle for mathematics education for nurturing critical mathematical literacy within a 21st-century society that fosters diversity. The mathematics education of the future requires both students and teachers to create shared values through SCOEPs.

2) In Japan, teaching practices based on the perspective of critical mathematics education and socially open-ended problems have been conducted for students from kindergarten to university (e.g., Nakawa & Kosaka, 2019; Shimada, 2019; Hattori & Fukuda, 2019; Inoue et al., 2018; Kubo, 2019). In addition, Hattori (2020) explored the same socially open-ended problem with 5th–6th-grade elementary school students (double-grade) and 2nd-grade middle school students and reported that the middle school students expressed more 'socially-oriented' critical thinking than elementary school students. Therefore, similar results may be obtained in SCOEPs.

CONFLICTS OF INTEREST

No potential conflict of interest relevant to this article was reported.

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