Original Article

Teacher Noticing for Supporting Student Proving: Gradual Articulation

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Abstract

Proof has been paid much attention in school mathematics and the literature. To date, the extant literature has focused more on difficulties in teaching and learning of proof rather than support that can help teachers to teach proof and develop student's understanding about proof. This study examined a teacher's practice that offers an explicit instruction of proof as part of everyday lesson. The results showed that the teacher supports student proving by way of teacher noticing with a clear definition of proof in the literature and that teacher noticing instructional practice needs to be more attuned for teaching proof. In particular, the teacher's moves for supporting student proving differ by what aspects of proof that the teacher pays attention to at a given time and by judgement on whether an argument qualifies as a valid proof. This instructional practice would provide a guidance and foundation for teachers to engage students in proving as part of everyday lesson.

Keywords: proof, reasoning, teacher noticing, teacher practice, enacted curriculum

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I. INTRODUCTION

Proof has been paid much attention in school mathematics and its importance has been well noted by scholars (e.g., Ball et al., 2003; Hanna, 1995; Knuth, 2002a, 2002b; Schoenfeld, 1994; Stylianides & Ball, 2008) and recommendations (CCSSO, 2010; Ministry of Education, 2005, 2015; NCTM, 1989, 2000). In stark contrast to attention paid to proof in school mathematics, the extant literature has documented the difficulties underlying teaching and learning proof (Knuth, 2002a, 2002b; Knuth et al., 2020; Senk, 1985). What makes instruction of proof dauntingly difficult is due in part to pre-service teachers' lack of understanding about what constitutes a proof (Coe & Ruthven, 1994; Harel & Sowder, 1998) and their limited perceptions of what proof does in mathematics (Basturk, 2010; Knuth, 2002a). For instance, when asked to judge whether an argument qualifies as a valid proof, teachers tend to pay excessive attention partially to form (rather than substance) or superficial (or even cursory) features of arguments rather than evaluating them holistically (Coe & Ruthven, 1994; Harel & Sowder, 1998; Kim, in review; Knuth, 2002b).

The issue of teacher's limited content knowledge about proof required to teach proof and pedagogical knowledge about students and proof coincide with students' not sophisticated understanding about proof and their performance on proof writing. In light of Senk's (1985) study, students' performance on proof writing did not seem promising at all given that a full year of geometry did not improve their performance. Chazan (1993) provided evidence that students have somewhat contradictory understanding about mathematical proof: Proof is evidence versus evidence is proof. As students' understandings about proof are not completely independent entity from teacher's knowledge and understanding about proof, support or guidance provided in written curricula might be of much help for teachers. However, Stylianides (2007) examined guidance offered for teachers in textbook with attention paid to proving-related tasks and reported that the vast majority (90%) of tasks were given without anything more than possible solutions. To address the aforementioned issues, there is a need for a guidance or foundation given for teachers to be better prepared to teacher proof.

The aim for this study is to report a teacher's pedagogical moves for teaching proof that provides teachers a baseline for what aspect(s) of proof they need to attend to when supporting student's proving so as to provide opportunities for teachers to accumulate knowledge about students as well as proof and for students to engage in proving in order to enhance understanding about proof and accordingly proof writing. In identifying instructional moves for supporting student proving, I first consider what constitutes a valid proof in a classroom community at a given time and moves that the teacher takes to support student proving and subsequently find to be promising the approach of teacher noticing (Sherin, Jacobs, & Phillipp, 2011) in that it helps teachers to be better at making at-the-moments decisions freshly and wisely (Mason, 2002) becoming sensitive and sensible to what students provide as response.

For the discussions that follow, some terms are worth clarification: argument, reasoning, proof and proving. By *argument*, I refer to a sequence of proposition, statements, or reasons to convince others by

demonstrating that a mathematical claim holds true or false. By reasoning, I refer to one's endeavor to provide reasons to justify her or his argument. By proof, I refer to mathematical argument that is readily accepted by the society of mathematicians. By proving, I refer to one's act to develop a proof including evaluating conjectures, getting insights into proofs, and writing proofs. Before turning our attention to next section, it is necessary to present the research questions that guide this study:

- a) What pedagogical moves may a teacher take to support student proving when the student's argument does not qualify as a proof?;
- b) What pedagogical moves may a teacher take in order to extend student's understanding beyond developing a proof?;

II. LITERATURE REVIEW

Despite emphases placed on proof across recommendations and curricula, students' and teachers' understandings of proof have been far from being promising. NCTM (2000) considers proof and reasoning as a fundamental aspect of student's learning of mathematics at all grade levels and states:

Instructional programs from prekindergarten through grade 12 should enable all students to: recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; and select and use various types of reasoning and methods of proof (p. 56)

In line with NCTM (2000)'s emphasis placed on proof, South Korea (Ministry of Education, 2015), Singapore (Ministry of Education, 2005), and US Common Core (CCSSO, 2010) uphold the value of proof in their curricula. However, Senk (1985) reported earlier that students' understandings of proof and their performance on writing proof did not reach at the level of mastery and that a full year of geometry did not seem to improve their performance on proof writing. In a similar vein, Schoenfeld (1986) claimed that student's understanding about proof remains not sophisticated enough to be able to distinguish empirical arguments (of which source of conviction about truth originates from examples) from proofs even by the end of K-12 education.

Such yet-to-be-sophisticated understandings about proof are not only the case for students but for teachers. Harel & Sowder (1998) investigated pre-service elementary teachers' proof schemes and reported that the pre-service teachers show limited understandings about proof such as evaluating mathematical arguments based primarily on form or appearance of the arguments and failure to distinguish mathematical proofs from empirical arguments. Knuth (2002b) documented that teachers tend to encounter difficulties in distinguishing arguments that qualify as proofs from those not. In particular, there was inadequacy in teachers' understanding about what constitutes a proof. Similarly, Ko & Knuth (2009) examined relationships between undergraduate mathematics majors' understandings of

continuity and their competence of producing proofs and counterexamples. The authors documented even undergraduate mathematics major students have difficulty producing proofs and providing counterexamples against mathematical arguments.

Lack of students' understanding about proof seems to be in part due to smatterings of opportunities for students to engage with proving-related activities. Bieda, Ji, Drwencke, & Picard (2014) noted that there is lack of opportunities available in textbook for students to engage with proving-related tasks. Furthermore, students' exposure to a variety of proof methods (e.g., *reductio ad absurdum*, mathematical induction, proof by contradiction) seems to be limited given the written curricula. In light of Begle's (1973) point that student learning is largely directed by text, this brings to the fore teacher's role in promoting proof as integral part of everyday practice in that it is teachers who enact the text with variations (Remillard & Bryans, 2004). Such variations in enactments of reasoning-and-proving related tasks might be influenced by teachers' views about proof.

Teachers' views about proof are limited in scope and accordingly judgement on proofs are made with overreliance on appearance. The limited views about proof are observed in form of equating empirical arguments with mathematically valid proofs (Balacheff, 1988; Coe & Ruthven, 1994; Knuth et al., 2020; Martin & Harel, 1989; Porteous, 1990) and searching for a warrant to justify the method used in a proof that is *extrinsic* to (i.e. not directly originated from) the proof (Harel & Sowder, 1998). One of ways how teachers search for an extrinsic warrant of a proof is identifying similarity between the appearance of a proof at hand and proofs that they have seen before (Coe & Ruthven, 1994; Harel & Sowder, 1998) being ritualistic (or even superficial) aspect of proof (Knuth, 2002a). This calls for a need to develop teachers' understanding about proof in general, their sensitivity and sensibility to what constitute a mathematically valid proof in particular. That is *desideratum* for teaching proof consistently and persistently across all grade levels. This study is intended to develop such sensitivity and sensibility for teachers by offering a foundation of what aspect(s) of proof they need to attend to when judging students' mathematical arguments and what action(s) teachers might take after making judgements about students' initial arguments. To that end, this study borrows wisdom from the literature in teacher noticing.

Teacher noticing has been paid increased attention in the field of mathematics education. Teacher noticing in mathematics education is of importance in that it enables one's better sensitivity to *attending to*, *interpreting*, and *responding to* student responses (Sherin et al., 2011). In literature, though much attention has been given to teacher noticing in the context of mathematics or particular subject areas (e.g., Walkoe, 2015), what lacks in the literature is that not many studies have paid particular attention to proof or instruction of proof taking a lens of teacher noticing. Even such studies focus less on *how* teachers teach how to prove rather than *what* teachers notice when they participate in professional development programs that place much emphasis on a subset or an element of proving-related activities (e.g., generalizing mathematical phenomena) rather than proving as a whole. This study draws distinction from the aforementioned studies in that the data of this study was based on the practice implemented in

classroom and the emphasis placed on this study is *how* rather than *what*.

In the literature, researchers use different terms for teacher noticing. For example, Sherin, et al. (2011) and Mason (2002) use the term *teacher noticing* while Goodwin (1994) and Sherin (2007) use *professional vision* and Jacobs, Lamb, & Philipp (2011) refer to *professional noticing* for similar constructs. What the frameworks share is the primary goal for teachers to get to become more sensitive to student mathematical thinking and better responsive to student responses so as to act *freshly* in a typical situation (Mason, 2002). The framework on teacher noticing that this study is based on is the work of Sherin et al. (2011).

The framework categorizes teacher's action into three labels: *attending to*, *interpreting*, and *responding* to student mathematical thinking (Sherin et al., 2011). By *attending to*, it refers to teacher's action that a teacher pays attention to student mathematical thinking and deems the mathematical thinking as worth addressing. By *interpreting*, it refers to teacher's action that a teacher interprets the student mathematical thinking to grasp what the student meant. By *responding to*, it refers to teacher's action that a teacher makes response to the student mathematical thinking after considering what might be needed to be addressed based on the interpretation one comes to have. The framework enables an explanation on how teacher's at-the-moment decision making occurs, however, what it falls short of is that it does not offer a guidance or basis for *what* teachers need to attend to, interpret and respond to at critical and fleeting moments in classroom. In this study, a fine-grained set of teacher's actions that would be taken will be delineated in the framework that offers a guidance for teachers who are not seasoned well in teaching proof supporting student proving.

In the literature, the context and the data of the studies are limited in that the studies take a lens of teacher noticing in teaching and learning proof. The extant study on instruction of proof with a lens of teacher noticing is based primarily on the data collected from professional development programs rather than classroom observations and teacher's reflection with much attention to what teachers notice or might have noticed (e.g., Jacobs et al., 2010; Melhuish, Thanheiser, & Fagan, 2019; Melhuish, Thanheiser, Fasteen, & Fredericks, 2015; Melhuish, Thanheiser, & Guyot, 2020). As such studies involve participant teachers' observations on other teachers' video-taped lessons and entail discussions that focus on what they notice while watching video recordings, the results of the studies only enable *extraspective* (i.e. of outsider's perspective) analyses or the meta-analysis of the precedent analyses without offering insight into *introspective* (i.e. of insider's perspective) narratives (Mason, 2002). What is needed in the literature is a construct originated from an insider (i.e. the instructor of the lesson) based on reflection of one's own teaching not merely experience itself. As Mason (2002) puts, "One thing we seem not to learn from experience, is that we rarely learn from experience alone" (p. 64). This study aims to identify teacher's pedagogical moves that support student proving and to report the results of a teacher's explicit instruction on proof and his reflection on the instruction. In particular, this study will draw connections between the extant literature on proof and suggest pedagogical moves that support student proving.

III. METHODS

1. The context

The study took place in a all-boys high school in urban area of a city located at the middle of Korean peninsula. The student participants were all enrolled in grade 10 (aged 15~16 years) and the instructor had 3 years of teaching experience by the beginning of the study. The study was conducted as a teaching experiment by the instructor with primary focus initially on problem posing and solving and he shifted emphasis more on reasoning-and-proving related activities that include making conjectures and developing proofs by leveraging ideas originated from observations on mathematically similar examples that can potentially be reduced to a conjecture. This way, students might develop conjectures that offer with students opportunities to develop proofs for the conjectures that they formulated. For consistency in form and persistence in keeping records, the instructor designed a worksheet based on “What-if?” (Brown & Walter, 1983).

The intent of the problem posing activity was two-fold: to promote student problem posing and solving; and to encourage students to make generalizations based on examples generated by way of problem posing as problem posing can lead to generalizations by varying a subset of the attributes of a problem (Brown & Walter, 1983; Walter & Brown, 1977). This activity provides students with opportunities to experience dynamic nature of proving-related activities (e.g., formulating conjectures, evaluating the conjectures with relevant examples into account, making revisions for the conjectures to have them to hold true) and proof writing that may ensue (Ellis et al., 2013; Lakatos, 1976; Stylianides, 2007b).

The problem posing activity was part of everyday lesson and discussions took place as needed. The activity began with student’s choosing of a problem to work on and solve the problem in writing. The instructor read and left comments for students to improve their proof writings. The comments were made to offer suggestions indirectly to point to what students should consider to gradually develop valid proofs or extend their understanding. The instructor’s intent of making indirect comments was to avoid making comments that might engender *Topaze effect* (Brousseau, 2006).

It would be remiss not to make explicit for the reader the shared definition of proof in the classrooms where the study took place. The definition of proof used in this study is:

Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

It uses statements accepted by the classroom community (*set of accepted state-ments*) that are true and available without further justification;

It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and

It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the class- room community. (Stylianides, 2007b, p. 291, italics in original)

Borrowing the definition of proof by A. Stylianides (2007b)¹⁾, students' proofs were judged in regard to: what statements were taken as *true*; what modes of reasoning were deemed to be *valid*; and what representations of reasoning were deemed to be *appropriate*. With regard to true statements, the instructor and the students decided to limit the extent to which statements were taken as true (i.e. neither being questioned about the truth of them nor asked to prove them) to those theorems that were included in the textbook prior to the time of citing them. If there were variations across the textbooks that students had used, limiting the extent of true statements would have been more daunting than it sounds for teachers and students. However, this was not the case since Korean mathematics textbooks are monitored and evaluated by the Korean ministry of education before use in school to make consistent the content and the depth of the content included in textbooks. With regard to *valid* modes of reasoning, although these were considered as student's prior knowledge in that all of them learned about proof at their grades 8 and 9 as prescribed in the national curriculum, the knowledge that they are expected to possess at the end of grade 9 was revisited and reiterated over the course of this study. Finally, with regard to *appropriate* modes of representations, the instructor considered representations as what students need facilitation and gradual articulation moving toward proofs since the instructor deemed this aspect of proof not to be necessarily reflecting student's thinking. These will be elaborated more in a later section in regard to teacher's actions involved in facilitating students' thinking toward developing proofs.

It should be noted that the instructor and the author are the same person while the author analyzed the data collected by the instructor five years later. Some might argue that the author should be situated as a participant-researcher, however, this is not the case for this study in that the author did not participate in designing the study and collecting data as the researcher at that time.

2. Data collection

The data corpus of the study consists of students' written work on proof writing, video-taped lessons (two 50-minute class periods), and teacher's comments on students' work. Given students' work was collected on a weekly basis and there were 105 students whom the teacher taught from 3 classrooms (each class occupied by 35 students), 105 worksheets were collected in most weeks (except for circumstances such as national holidays, mid-terms and finals, or field trips). The number of the worksheets at the end of

1) A. Stylianides (2007a) examined the notion of proof in elementary school by analyzing students' arguments that are potentially proofs and he (2007b) further developed the definition of proof drawing upon theoretical frameworks on proof and his previous work (2007a).

the academic year was about 2,000 and I excluded some of those that do not involve student proving or teacher's feedbacks that are pertaining to the focus of the study. Finally, I took a sample from the set of students' work (after the exclusion was made) and analyzed the sample of 300 which accounts for 20%.

3. Data analysis

The pedagogical moves that I will report in the following section were based first on the instructor's reflection on his instruction that places emphasis on student's formulating conjectures and developing proofs and on analyses of students' written assignments along with teacher's feedbacks on them. In light of student's written assignments and the instructor's feedbacks, the teacher's actions were identified and categorized with regard to: whether an initial argument constitutes a proof; and what aspect(s) the teacher attended to at a given time towards prompting student to make revision(s) to make the initial argument qualified as a proof. This process of identification of cases pertaining to the foci of this study was reiterative in nature and initial emerging codes were gradually refined and subsumed to the resulting framework.

IV. RESULTS

In this section, I will provide pedagogical moves with particular attention to teacher's decision making in what teacher's action follows after the teacher attends to student's initial argument. This framework provides a foundation for teachers to interpret and responding to student's initial argument that A. Stylianides (2007b) might call *base argument*. Then teacher's actions are categorized into two categories in regard to what purpose of an action is intended by the teacher: *generative* and *corrective* actions. Finally, sub-actions under the aforementioned categories will be introduced.

The sub-actions are taken based on teacher's judgement on student's initial argument and ensuing decision making at a given time. This should be noted that the reason why I use "at a given time" is that the aspects of proof (true, valid, and appropriate) are not consistent over time. For example, a classroom community's set of true statements accumulates as time goes. As depicted in Figure 1, what the teacher did in the first place was judging whether an initial argument provided by a student constitutes a proof. To make a judgement of whether an initial (or ensuing) argument constitutes a proof, a teacher pays attention to three aspects of proof: *true*, *valid*, and *appropriate* (Stylianides, 2007b). If all the aspects of the initial argument meet the criteria for a proof (i.e. reach or surpass the proof threshold), then the initial/ ensuing argument will be deemed to be a proof. In that case, the teacher took *generative* actions that might reinforce student's understanding by expanding the domain of the initial/ensuing argument, posing mathematically similar questions to ponder on, or offering opportunities for the student to apply the proof method used in the initial argument to other similar circumstances. Otherwise, *corrective* actions were

Figure 1 Interpreting and Responding to student's initial argument (Note that what is inside the dashed rectangle depicts the initial stage when a student presents an initial argument)

taken in order for the teacher to encourage a student to make necessary revisions to make the argument qualify as a proof. In making this series of actions, the teacher set what Stylianides (2007b) called *proof threshold* and he made judgements based on set proof threshold with regard to the three aspects of proof: true, valid, and appropriate. With proof threshold taken into consideration, possible didactical situations can be depicted as below (Figure 2).

For example, the farthest left one on the first row depicts a situation where student's initial argument does not qualify as a valid proof due to part of the argument that does not surpass or reach the proof threshold set in regard to true aspect of proof. Note that I exclude two possible cases in Figure 3: one that does not meet any aspect of proof threshold thus all aspects should undergo revision to qualify as a valid proof; and the other that meets all the aspects of proof threshold.

Teacher's decision making occurs as depicted in Figure 1. Though the figure seems to have the only ensuing judgement that follows after the initial phase of teacher's judgement and subsequent decision making, this kind of teacher's decision making based on the initial and ensuing judgements takes place where ensuing argument does not qualify as a valid proof. In cases where generative action ensues, generative action is taken more than once in order to push forward student's understanding even further.

Figure 2 Possible didactical situations (Gray bars represent the aspect(s) of student's initial/ensuing argument that doesn't reach or surpass the proof threshold)

Figure 3 The sample of student's work (Note: it is partially blurred and erased out due to copyright issues)

Thus, teacher's actions following teacher's judgement of whether an argument qualifies as valid proof are reiterative in nature: not dead ends but open doors for what would come next.

Corrective actions taken by the teacher are divided into what aspect of student's initial argument the teacher attends to or judges there is a need to further revise for the argument to qualify as a valid proof. The actions are shown in the *Table 1*. Attending to true aspect of a proof, corrective actions ensue: ask to specify the reference of a statement/assumption included in a proof or prompt to prove a conjecture/assumption used in a proof. These actions are taken to question what ground(s) a student considers statements to hold true: either specifying references or providing proofs. Attending to valid aspect of a

Table 1. *Corrective actions*

Aspects of proof	Corrective actions
True	Prompt to specify the reference of a statement/assumption included in a proof
	Prompt to prove a conjecture/assumption used in a proof
Valid	Prompt to justify the proof method used in a proof
	Prompt to provide reasoning in a proof
Appropriate	Request to express what a student intends to convey
	Prompt to restate/articulate what a student intends to convey
	Ask to use representations/expressions that are relevant and within a student's cognitive reach
	Ask to refer to and use the representations/expressions used in the known sources such as textbook(s)

proof, corrective actions ensue: prompt to justify the proof method used in a proof or prompt to provide reasoning in a proof. These actions are taken to solicit justifications on the proof method (e.g., proof by contradiction, proof by contrapositive) used in a proof as a whole or the logical chain(s) of reasoning as part of a proof. Justifications expected by these kinds of corrective actions need not be proofs but explanations about why the proof method in a proof is valid. Finally, attending to appropriate aspect of a proof, corrective actions ensue: first, request to express what a student intends to convey; second, prompt to restate or articulate what a student intends to convey; third, ask to use representations/expressions that relevant and within a student's cognitive reach; lastly, ask to refer to and use the representations/expressions used in the known sources such as textbooks. The first two actions are taken to solicit student's intended meaning which might not be conveyed well enough to be understood by the reader or the audience and clarify the meaning while the last two actions are taken to promoting use of representations that are consistent with those acceptable in mathematics. So, the four actions under appropriate aspect of proof serve as gradual modification from student's plain language to mathematically acceptable expressions and representations (cf. Kim, 2020).

The teacher did not place emphasis on correctness of representations or expressions used in initial arguments. The teacher's intent was aligned with Zack (1999)'s point that, though often betrayed by everyday language, children's thinking shows complex mathematical structures. Consistent with Moschkovich (2010)'s emphasis on focusing on student's reasoning and supporting student reasoning rather than accuracy in language that conveys the reasoning, his assumption was that students might lack expressions or representations that are readily acceptable by mathematicians albeit their underlying reasoning is mathematically valid. That is part of reason why corrective actions under appropriate are more fine-grained to elicit student's reasoning rather than and prompt students to express what they intend to convey at a given time gradually increasing in clarity and correctness that are in proximity to what mathematicians use.

V. DISCUSSION AND CONCLUSION

Proof has been paid much attention in the literature and its importance in school mathematics has well been emphasized in curricula (Ball et al., 2003; Knuth, 2002a, 2002b; NCTM, 1989, 2000; Schoenfeld, 1994; Tall, 1999).

The activity, as part of this study (see Kim, 2020 for detail), is an instructional practice that is closely aligned with NCTM's standpoint (see NCTM, 2000, p. 56 for detail) toward proof and reasoning as mentioned above in that the activity involves making and evaluating mathematical arguments and student's discretion to choose proof methods that necessitate various types of reasoning. Furthermore, as it is neither specific to subject nor limited to content, the activity is readily enacted in class. The pedagogical moves and the proof writing activity can help teachers to better organize classroom discussions that are not orchestrated by all teachers (Stylianou & Blanton, 2011). My hope for the reader is that the moves are found to be a stepping stone for teachers who are early in their teaching career toward the goal "proof for all" and thus to be open to modifications and adaptations.

The distinction between the pedagogical moves reported in this paper and one that Stylianides (2007b) proposed should be made. While Stylianides (2007b)'s work on a definition of proof in school mathematics and instructional practices that cultivate proof and proving has drawn much attention in the literature, his work assumes a classroom as a whole rather than a community of students and the instructional practices (or, *alternative courses of actions* that a teacher might take) in the study are less specific in nature than those contained in this study. The pedagogical moves in this study do not assume that a teacher's role in facilitating student proving should be active as Stylianides (2007b) argued: rather, in this study, teachers are assumed to be supporters who pose questions and help students to articulate their reasoning and have them make progress toward writing proofs. What is common in both works seems that it is teachers who cultivate proof and proving in classroom as everyday practice.

The value of this study is that the pedagogical moves provide more opportunities for students to engage in proving-related activities than those available in textbook: exploring particular examples, formulating conjectures, evaluating conjectures, generating propositions, attempting to prove the propositions, evaluating and critiquing proofs. The proving-related activities are in resonance with how mathematicians use examples when developing proofs (Ellis et al., 2013; Lockwood et al., 2013). They can provide opportunities for students to experience what proof affords in mathematics as mathematicians do. The importance of providing such learning opportunities lies on student's perception about proof and its affordance in mathematics. As Schoenfeld (1994) puts,

There are, I think, three roles of proof that need to be explored and understood: the unique character of certainty provided by air-tight mathematical arguments, which differs from that in any other discipline and is part of what makes mathematics what it is; the fact

that proof need not be conceived as an arcane formal ritual, but can be seen as the mere codification of clear thinking and a way of communicating ideas with others; and the fact that for mathematicians, proving is a way of thinking, exploring, of *coming to understand* and that students can and should experience mathematical proving in the same ways. (p. 74, italics in original)

This study suggests a mathematically rigorous and meaningful practice for students to engage in proving that is often deemed to be ritualistic. In light of Knuth (2002b)'s study, even in-service teachers tend to consider reasoning and proof as ritualistic rather than fundamental in mathematics and geometry as the place where proof is explicitly and intensively treated during K-12 education and the proof method contained in the subject is limited to direct proof which involves chains of reasoning from the antecedent to the consequent of a proposition. These kinds of limited experiences available in curricula engender limited views on and perceptions about proof for both teachers and students (Balacheff, 1988; Coe & Ruthven, 1994; Harel & Sowder, 1998). One way to resolve this issue is providing learning experiences that convey personal meaning and explanatory power to students. As Schoenfeld (1994) pointed out, "In most instructional contexts proof has no personal meaning or explanatory power for students. [...] Students believe that proof-writing is a ritual to be engaged in, rather than a productive endeavor" (p. 75). Davis (1985) made a point that demonstrating complete proofs obscures its social aspect of developing a proof. The results showed that discussions about proof and proof writing can be integrated into everyday lesson and social or personal needs (e.g., call for explanation, convincing others) of proof in class arise naturally thus leaving proof rich in meaning for students and possibly providing opportunities to experience different kinds of proof methods (e.g., *reductio ad absurdum*, mathematical induction, proof by contradiction) and consider proof to be a productive endeavor. To enact the pedagogy well, teacher's content knowledge about proof is particularly important.

Teacher's knowledge about proof plays a critical role in implementing the instructional practice proposed earlier in the study. As reported earlier, this is due, in large part, to the fact that judgement on whether an argument qualifies as a proof is made by the teacher. Though some might argue that students might be delegated authority as judges who evaluate arguments and assist each other in articulating their mathematical thinking and writing proofs, I have reservation to argue that it is unrealistic or difficult to do so. Rather, I would be cautious about doing so before students come to a consensus about what it means to be a proof and how they should make judgements on their own or others' arguments and classroom community constitutes class norms which value attempts rather than correctness, substance of argument rather than form or appearance of it, and help each other to fill gaps between their initial arguments and proofs. Without such norms, any arguments that do not qualify as proof in the first place are likely to be ceased to be invalid proofs and put a dead-end to the arguments leaving students with no opportunity to learn about proof and understand how a proof is developed after numerous attempts and subsequent

revisions.

Teacher's positioning students and assumptions toward a practice and students need further examination. The instructor involved in this study positioned students as active producers of mathematical knowledge and the pedagogy was based on the assumptions that transition to formal proof is a huge cognitive struggle for students (Tall, 1999) and students' reasoning might not well be appreciated when they lack language to explain their thinking and proof (Moschkovich, 2010; Zack, 1999). Future research needs to examine how teacher's positioning student as either producer, consumer, or elsewhere on a continuum with the two extremes impacts students' learning proof in regard to what support(s) teacher provides for students. In such study, some issues might arise to varying degrees when implemented in classrooms occupied by different student bodies, using different curricula, being in a different culture, and so forth. This also suggests some possible directions in the future research.

CONFLICTS OF INTEREST

No potential conflict of interest relevant to this article was reported.

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